CASIO

Learning Mathematics with ClassWiz

fx-570EX *fx-991EX*

Barry Kissane

CLASSWIZ

Support Classroom with Technology

CASIO Worldwide Education Website

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Reliable & Durable

PREFACE

Over 40 years, the scientific calculator has evolved from being a computational device for scientists and engineers to becoming an important educational tool. What began as an instrument to answer numerical questions has evolved to become an affordable, powerful and flexible environment for students and their teachers to explore a wide range of mathematical ideas and relationships.

The significant calculator developments of recent years, together with advice from experienced teachers, have culminated in the advanced scientific calculators, the CASIO fx-991 EX and the CASIO fx-570 EX, with substantial mathematical capabilities, including a spreadsheet, and extensive use of natural displays of mathematical notation.

This publication comprises a series of modules to help make best use of the opportunities for mathematics education afforded by these developments. The focus of the modules is on the use of the *ClassWiz* in the development of students' understanding of mathematical concepts and relationships, as an integral part of the development of mathematical meaning for the students. While meeting the computational needs of students throughout secondary school, and beyond, the *ClassWiz* can also be used to advantage by students to support their initial learning of the mathematical ideas involved; the calculator is not only a device to be used to undertake or to check computations, after the mathematics has been understood.

The mathematics involved in the modules spans a wide range from the early years of secondary school to the early undergraduate years, from early ideas of number and algebra through to the study of calculus, probability and statistics, as well as advanced topics such as vectors, matrices and complex numbers. I expect that readers will decide which modules suit their purposes. Although mathematics curricula vary across different countries, I am confident that the mathematical ideas included in the modules will be of interest to mathematics teachers and their students across international boundaries.

The text of each modules is intended to be read by both teachers and students, to understand how the *ClassWiz* is related to various aspects of mathematics, and also to help them to use it efficiently. Each module contains a set of *Exercises*, focusing on calculator skills relevant to the mathematics associated with the module. In addition, a set of exploratory *Activities* is provided for each module after the first, to illustrate some of the ways in which the calculator can be used to explore mathematical ideas through the use of the calculator; these are not intended to be exhaustive, and I expect that teachers will develop further activities of these kinds to suit their students. The *Notes for Teachers* in each module provide answers to exercises, as well as some advice about the classroom use of the activities (including answers where appropriate). Permission is given for the reproduction of any of the materials for educational purposes.

The material in many of these modules draws significantly on materials developed earlier by Marian Kemp and myself, refining and extending those materials to take advantage of the many innovative features included in the *ClassWiz*. I am grateful to CASIO for supporting the development of these materials, and appreciate in particular the assistance of Mr Yoshino throughout the developmental process, as well as the careful proof-reading of Mr Rabieh Al Halabi.

I hope that users of these materials enjoy working with the calculator as much as I have enjoyed developing the materials and I wish both teachers and their students a productive engagement with mathematics through the use of the *ClassWiz*.

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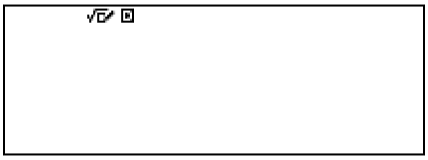
Table of Contents

Module 1
Introduction to *ClassWiz*

The CASIO *ClassWiz* calculator has many capabilities helpful for doing and learning mathematics. In this module, the general operations of the calculator will be explained and illustrated, to help you to become an efficient user of the calculator.

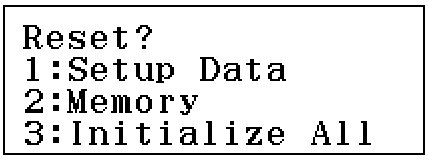
Entering and editing commands

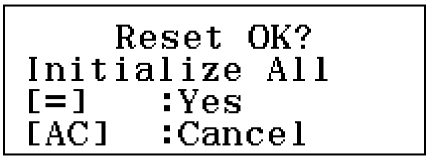
We will start with some computations. After tapping the [ON] key, tap [MENU] [1] to enter Calculate mode. The screen will be blank and ready for calculations, as shown below.

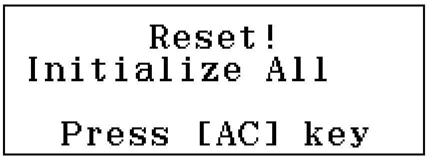


It is always wise to be aware of the calculator settings when using the calculator. Notice the two small symbols showing on the top line of the screen, even before you enter any calculations. The D symbol shows you that the calculator assumes that angles are measured in degrees. The other symbol shows you that the calculator has been set to accept and display calculations in natural mathematical notation (described by the calculator as *Math* mode).

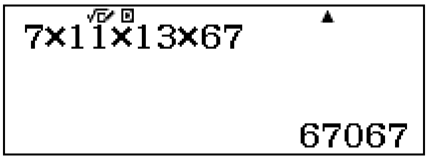
Each of these settings can be changed, as we will explain below. If your calculator looks different from this, it may be wise to reset it for now, until you are familiar with how to change settings. Tap [SHIFT] followed by [9] (don't hold down the [SHIFT] key) and follow the instructions to *Initialize All* by tapping [3], as shown below.

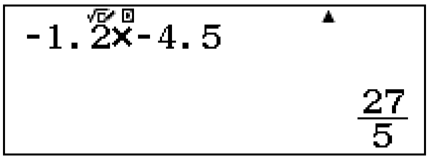




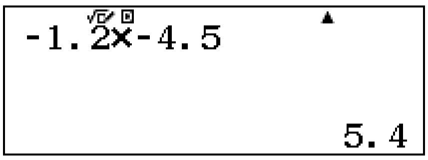


Calculations can be entered directly onto the screen in a command line. Complete a calculation using the [=] key. Here are two examples:

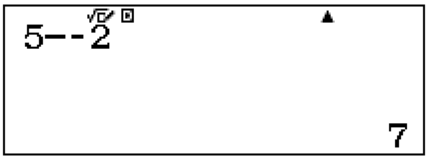




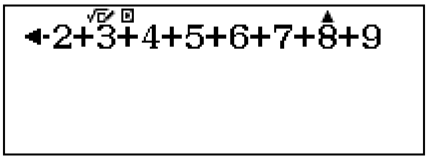
Notice in the second example that the result is showing as a fraction. You can change this to a decimal if you wish by tapping the [S⇔D] (Standard to Decimal) key. An alternative is to tap [SHIFT] before tapping [=], which will produce a decimal result immediately (as indicated by the symbol ≈ above the [=] key). Notice that no unnecessary decimal places are shown: the result is shown as 5.4 and not 5.40, which students sometimes obtain with hand methods of calculation.

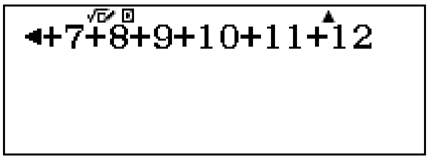


You should use the (−) key to enter negative numbers, as in the screen above. The − key is for subtraction. Look carefully at the screen below to see that a subtraction sign is longer than a negative sign.

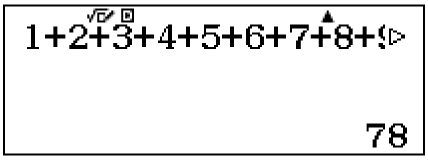


If a command is too long to fit on the screen, it is still acceptable for entry, as the screens below show. The command entered is to add the first twelve counting numbers.



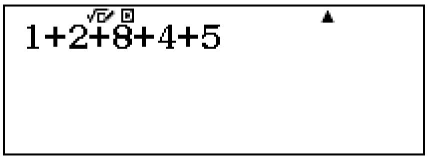


As you can see, the calculator automatically shows arrows on the display when a command is longer than the display. If you need to check or edit what has already been entered, you can move backwards and forwards with the two cursor keys ◀ and ▶ (These keys are on opposite sides of the set of four cursor keys key at the top of the keyboard.). Note especially that if you tap ▶ when the cursor is at the right end of the display, it will jump to the left end; similarly, if you tap ◀ when the cursor is at the left end of the display, it will jump to the right end. Tap = at any stage to calculate the result.



As shown above, it is not necessary to return the cursor to the end of the display before tapping =. Notice that only the first part of the command is shown, although the clear arrow indicates that there is more to be seen.

If you make an error when entering a command, you can erase the entire command and start again using the AC key or you can edit the command using the DEL key. Position the cursor to the right of a character and tap DEL to delete a single character. You can then add another character by entering it from the keyboard.

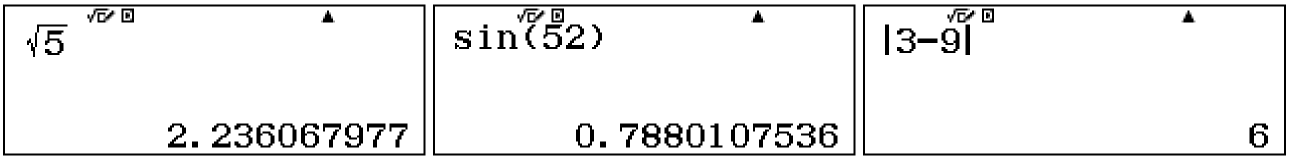


Characters can also be inserted using SHIFT DEL, but it is generally not necessary to do this. There is also an UNDO command, allowing you to undo the last operation. Tap ALPHA DEL Try these for yourself by entering the above command and then editing it to replace the 8 with a 3 before you tap =.

Mathematical commands

Many special mathematical operations are available on the calculator. These will be explored in some detail in later modules, so only a brief look at a few of the keyboard commands is provided here.

When in Math mode, you can usually enter a mathematical command in the same way in which you would write it, as the calculator uses natural display. In some cases, the keyboard command is the first one you need to use, while in other cases, a command is entered after the number concerned. Here are three examples for which the command key is entered *first*:

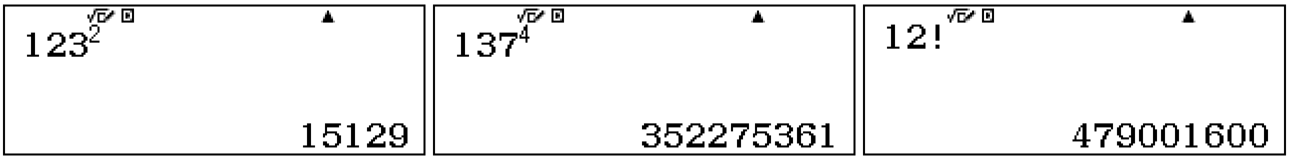


The square root key [√■] is discussed in more detail in the next module. Notice that in the example above, [SHIFT] [=] was used to get a numerical approximation in the form of a decimal.

Notice that the sine command [sin] has been completed with a right parenthesis. Although this is not strictly necessary here, as the calculator will compute the value without it, it is a good practice to close parentheses, rather than leave the calculator to do it for you. As the calculator is set to degrees, the command gives an approximation to sin 52°. This and other trigonometry keys are discussed in more detail in Module 5.

The absolute value command requires two keys, [SHIFT] and [(], and is represented on the calculator keyboard with *Abs* (written above the [(] key). The example above shows that the distance between 3 and 9 is 6, ignoring the direction or sign of the difference.

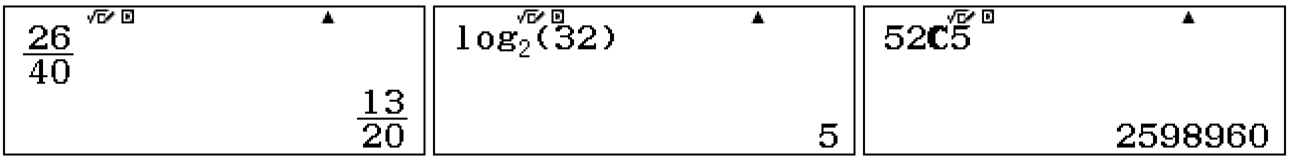
Here are three examples for which the command key is used *after* the number has been entered:



Tapping the [x^2] key after entering a number will give the square of the number; in the example shown above, 123 x 123 = 15 129. Most powers require the use of the [$x^■$] key, as shown to find the fourth power of 137 above. The mathematics associated with these keys is discussed in more detail in the next module.

The factorial command *x!* is needed to calculate 12 factorial, which is 12 x 11 x 10 x … x 1. This is the number of different orders in which twelve things can be arranged in line: 479 001 600. The mathematics associated with factorials is important in probability, and so is discussed in detail in the Probability module, Module 12. To insert the command, tap [SHIFT] and then the [x^{-1}] key.

Some mathematical commands require input of more than one number. In general, when using natural display mode, you should enter these commands in the calculator in the same way in which you would write them by hand. Here are three examples for which more than one input is needed:

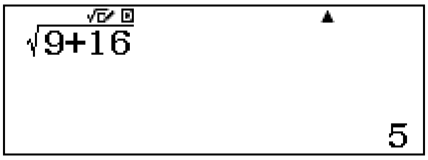


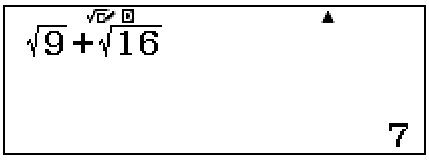
The first example above shows a fraction being entered. You can either tap the [■/■] key first and enter the numerator and the denominator, using the cursor keys such as [▼] and [▲] to move between these, or you can start by entering 26 and then tap the [■/■] key. The use of fractions will be addressed in more detail in Module 2. Notice that mixed fractions (■■/■) can be entered with [SHIFT] [■/■].

The second example above shows the use of the calculator to find the logarithm to base 2 of 32: that is, the power of 2 that is needed to obtain 32. Tap the [log▪□] key first and then enter both 2 and 32 as shown, using the ▶ and ◀ keys to move between these. The mathematical ideas of logarithms are explored in more detail in Module 6.

The third example above shows the number of combinations of 52 objects taken five at a time, represented in mathematics by $_{52}C_5$. The best way to enter this into the calculator is to first enter 52, then use the nCr command with [SHIFT] [÷] and then enter the 5. The result in this case shows that there are 2 598 960 different five-card hands available from a complete deck of 52 playing cards. Calculations of these kinds are discussed in more detail in Module 12.

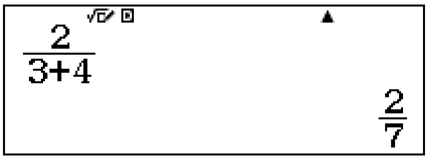
When using mathematical commands in calculations, it is often necessary to use the cursor key ▶ to exit from a command before continuing. The cursor will remain in the same command until moved outside it. To illustrate this idea, study the following two screens carefully.

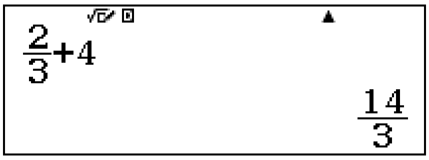




In the first case, after tapping [√▪], we entered [9] [+] [1] [6] and then tapped [=]. The cursor remained within the square root sign. In the second case, after entering [√▪] [9], we tapped ▶ to move the cursor out of the square root before tapping [+] [√▪] [1] [6] to complete the command and then [=] to get the result. Try this for yourself to see how it works.

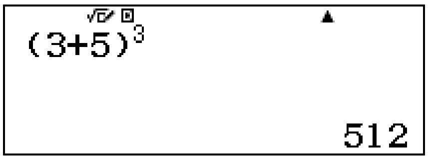
Here is another example of the same idea, using fractions:

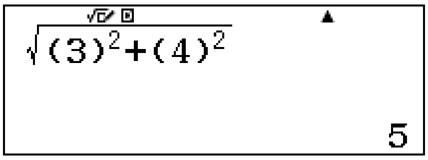


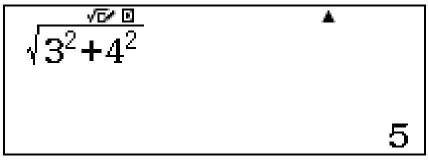


In the second of these screens, ▶ was used to exit the fraction before adding 4.

Of course, it is always possible to use parentheses to clarify meanings in a mathematical expression, but it is not always necessary. For example, in the three screens below, the parentheses are necessary in the first case, but not in the second, as the third screen makes clear.



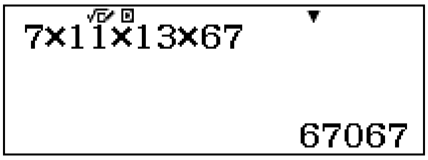


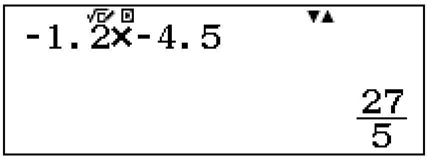


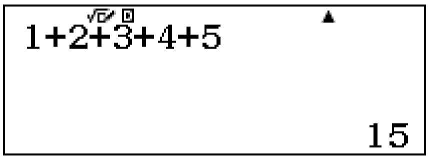
As it takes longer to enter expressions with parentheses, it is a good idea to develop expertise in constructing expressions without them, when possible.

Recalling commands

You may have noticed a small upward arrow at the top of your calculator display, on the right. This indicates that you can use the ▲ cursor key to recall earlier commands from a list. When you are at the top of the list, the arrow points downwards to show this. When between top and bottom of the list of recent commands, both up and down arrows will show, to indicate that you can recall commands in either direction. These three possibilities are shown below.

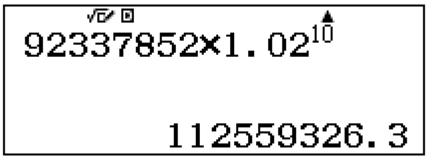




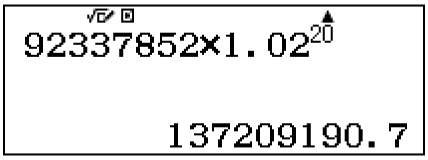


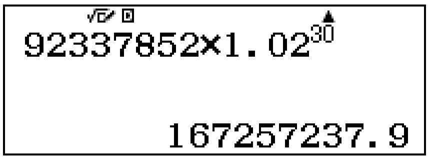
If you move the cursor key to display a previous command, you can enter the command again by tapping ⊜, or can edit it first (using ◀, ▶, DEL) and then tap ⊜. This is a good way of performing several similar calculations in succession, without having to enter each complete command again.

If you want to edit only the most recent command, there is an even easier approach. You need merely tap the ◀ key and then edit the command directly. For example, the following screen shows an estimate of almost 113 million for the 2020 population of the Philippines, which had a population of 92 337 852 in the 2010 census, and a population growth rate of 2% per annum. Notice that a growth rate of 2% can be calculated by multiplying a number by 1.02.

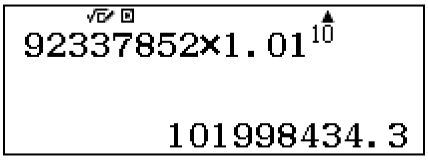


The easiest way to obtain an estimate for later years, assuming the annual population growth rate stays the same, is to tap ◀, edit the command to change the exponent of 10 to a different number and then tap ⊜ to finish. The screens below show the results for 2030 and 2040.





The very high growth rate of the Philippines in 2010 will lead to a population of more than 167 million in 2040. The same calculator process could be used to predict the population if the growth rate was assumed to be reduced drastically from 2% to 1%, as shown below, where the number of years as well as the growth rate have both been edited.

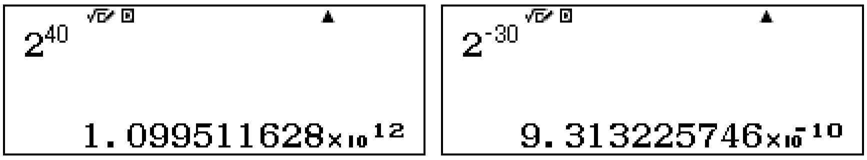


As you can see the population of the Philippines is estimated to be almost 102 million in 2020, if the growth rate were to be reduced to 1%, a figure around 10 million fewer than predicted for a growth rate of 2%. Successive predictions of these kinds can be made efficiently in this way, without needing to enter long and complicated expressions more than once.

The list of commands will be erased when you turn the calculator off, or change modes (as described below) but will not be erased when you tap the AC key, so it is wise to keep the calculator in the same mode and switched on if you think it likely that you will need some of the same sorts of calculations repeatedly.

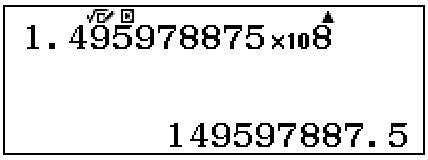
Scientific and engineering notation

When numbers become too large or too small to fit the screen, they will automatically be described in *scientific notation*, which involves a number between 1 and 10 and a power of 10. The precise way in which this happens depends on the decimal number format, which is described later in this module. To illustrate, the screen below shows two powers of 2 that require scientific notation to be expressed.



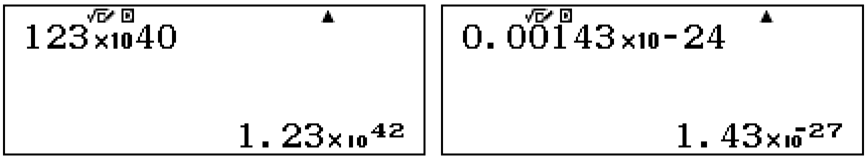
The precise value of the first result is 1 099 511 627 776, which does not fit on the screen, so it has been approximated using scientific notation. Notice that the last digit has been rounded upwards. Similarly, the second result has been approximated from 0.000000000931322574615479… to fit the screen.

Numbers can be entered directly into the calculator using scientific notation. Start with the number between 1 and 10, tap the [x10^x] key and then immediately enter the power of 10. For example, the average distance from the Earth to the Sun is 1.495978875 x 10^8 km, which can be entered in scientific notation as shown on the screen below.

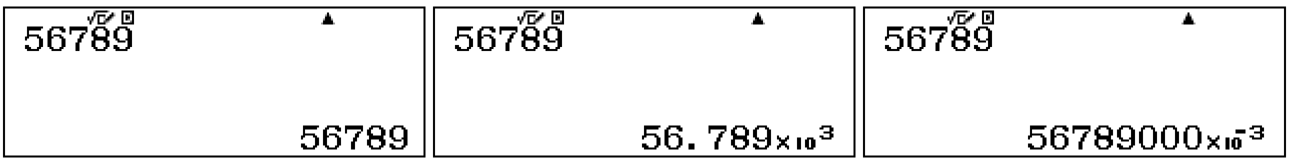


Notice that the exponent of 8 is not shown as raised on the screen, although it is interpreted by the calculator as a power. In the present mode used for display of results, notice that the calculator does not regard this number as large enough to require scientific notation, and so it is represented as a number, indicating that the sun is on average about 149 597 887.5 km from the earth – an average distance of almost 150 million kilometres.

Scientific notation requires the first number to be between 1 and 10. So, if you use the [x10^x] key to enter a number in scientific notation incorrectly (i.e. using a number that is not between 1 and 10), the calculator will represent it correctly, as the screen below shows.



Engineering notation is a different way of interpreting large and small numbers, using scientific notation with powers of 10 that are multiples of 3. This is convenient in many practical applications involving measurement, since units often have different names with such powers. For example, a distance of 56 789 m can be interpreted as 56.789 km or as 56 789 000 mm, as shown on the screens below, by converting numbers through successively tapping the [ENG] key.

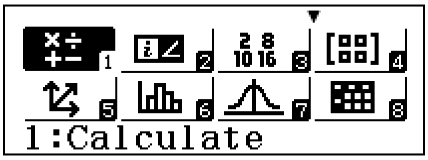


The number itself is not changed by these steps, but its representation is changed to make it easier to interpret.

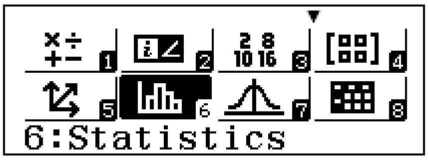
Notice also that conversions can be done in the opposite direction using [SHIFT] [ENG] (←).

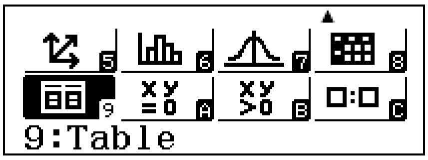
Calculator modes

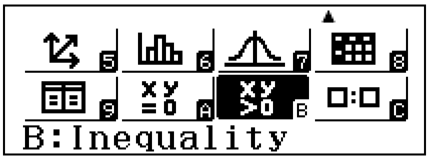
So far, we have used the calculator only for computations. However, the calculator can be used to explore many other aspects of mathematics, which are accessed in various *modes*. To see the choices, tap the MENU key, to get the screen shown below, which shows the first eight modes, beginning with the Calculate mode that we have been using so far.



You can move the cursor to highlight different modes and tap the = key to access the highlighted mode. The screens below show three more modes, with their descriptive icons.







The twelve modes available can be accessed by tapping MENU and the associated number or letter key. All of these modes will be explored in detail in later modules. For now, note the following brief overview of the other modes.

Mode 2: Complex mode deals with complex numbers, used in advanced mathematics. An example of a complex number is $\boldsymbol{i} = \sqrt{-1}$. When in Complex mode, the number $\boldsymbol{i}$ can be entered into the calculator and used for calculations. In Complex mode, special complex number operations are also available. Complex mode is used extensively in Module 9.

Mode 3: Base N mode allows you to undertake computations in different number bases as well as the usual decimal number base. The calculator keyboard has commands (written in blue) for converting numbers between binary, octal, hexadecimal and decimal number bases. These are especially important for computer science, as these bases are commonly used in computers. We will explore their use in Module 9.

Mode 4: Matrix mode is for matrix computations. You can define and use up to four matrices each up to 4 x 4 in dimensions and perform arithmetic and important matrix operations with them. We will use this mode extensively in Module 7.

Mode 5: Vector mode allows you to define and use up to four vectors with dimensions 2 or 3. Vector operations are then accessible. We will rely on this mode in Module 8.

Mode 6: *Statistics* mode is for various statistics, both univariate and bivariate, which are dealt with in the two respective Statistics modules, Modules 10 and 11. In this mode, various statistics calculations can be undertaken and a range of bivariate models explored.

Mode 7: *Distribution* mode is to access probability distributions, which will be addressed in Module 12. Normal, binomial and Poisson distributions are all addressed in this mode.

Mode 8: Spreadsheet mode allows you to define and to use spreadsheets for mathematical purposes. We will explore aspects of these in Module 13 and elsewhere.

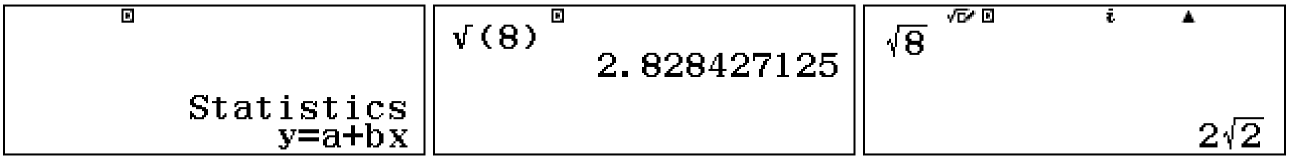
Mode 9: Table mode is for making tables of values of functions, which is useful for many purposes including sketching graphs and solving equations. We will use this mode in several modules concerned with functions and equations.

Mode A: Equation mode is for solving equations of various kinds. (Notice that the letter A is written on the keyboard in red, but requires only the (−) key to be tapped after MENU.) In particular, quadratic, cubic and quartic equations can be solved, as well as systems of two, three or four linear equations. We will use this mode extensively in Module 4.

Mode B: Inequality mode is for solving quadratic, cubic or quartic inequalities of various kinds. We will use this mode in Module 4.

Mode C: Ratio mode is for solving proportions, involving a pair of ratios. These are special kinds of equations that will be considered in Module 4.

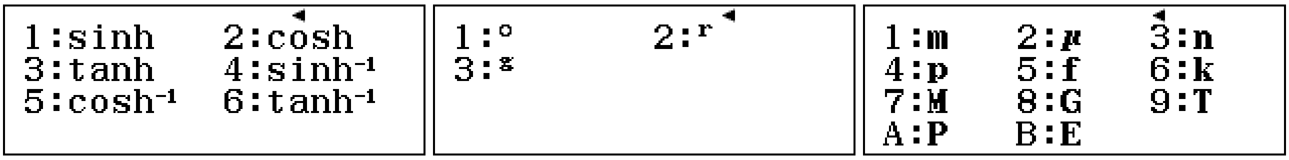
It is generally clear from your activity which mode the calculator is set in. If it is not clear, you can tap the MENU key (twice) to display and then remove the mode name. In some modes, the mode will be displayed when you tap the AC key, as shown below for Statistics mode. The screen on the right shows a small symbol ***i*** at the top of the screen to indicate that the calculator is in Complex mode.



While computations can still be performed in these modes, you will not always have the benefits of natural display notation on the screen, as shown in the middle screen above in Statistics mode, so it is generally better to use Calculation mode if your main purpose is doing calculations. In this Introductory Module, we suggest that you keep your calculator in Calculation mode. Notice that the calculator screen memory is cleared whenever you shift modes.

OPTN commands

The OPTN key provides a menu of further options that are relevant to a particular mode and not available directly on the keyboard. For example, in Calculation mode, it provides access to the hyperbolic functions, to symbols for different angle measures and to various Engineering symbols to facilitate data entry, as illustrated below.

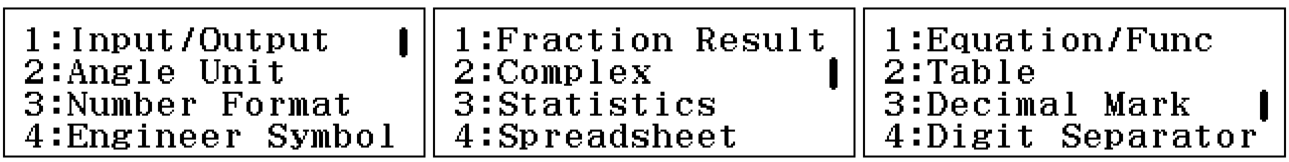


Similarly, in Statistics mode, the OPTN key provides access to various numerical statistics. We will explore the particular capabilities of these commands in later modules, as they become relevant.

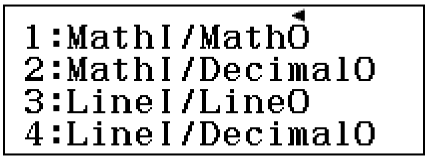
SET UP

We suggest that you now change back to Calculation mode by tapping MENU 1.

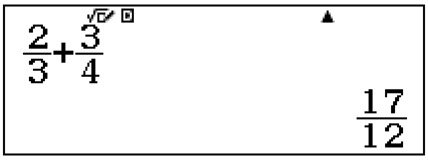
In any mode, the calculator can be set up in various ways by accessing SET UP (via SHIFT MENU). When you do this, you will notice that there are several screens in the SET UP menu, and an associated scroll bar on the right of the screen. You can move between screens using the ▼ and ▲ cursor keys and select a SET UP screen using a number. Here are the first three screens:

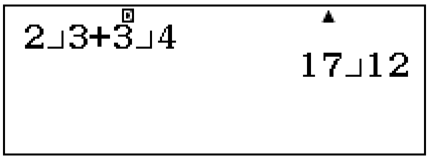


Input/Output
Tap 1 to see the four options available. The calculator display can be set up for input in either natural display (*Math*) format or single line (*Line*) format by tapping an associated number key:



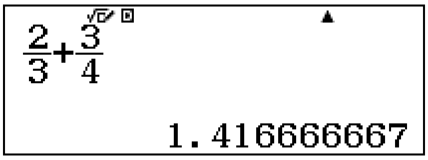
Math input formats (*MathI*) allow for various mathematical expressions to be entered in the conventional way; it is usually better to use these. In Line input formats (*LineI*), it will often be more difficult to enter commands, as they are restricted to a single line. For example, the following two screens show the same information, the first entered in Math format and the second in Line format. As well as looking different, although the same fraction key is used for the fractions, it is slightly more difficult to enter the fractions in Line format, as the numbers need to be entered in precisely the same order as they are written.

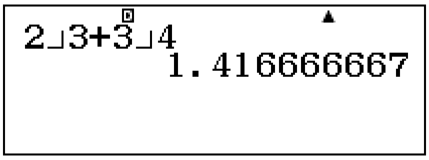




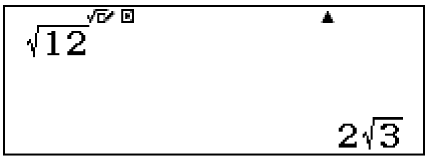
Exact results will not usually be shown in Line output format and the symbols will look a little different (as they do above).

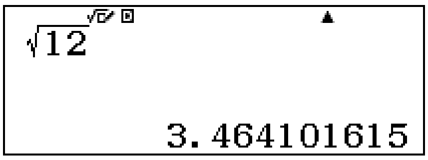
Sometimes, people like results of calculations to be represented as decimal approximations rather than exact numbers (although this is always possible using SHIFT = or the S⇔D key). If you prefer to do this, you should choose a *Decimal* output format for results. To illustrate this, the two screens below show the same calculation in *MathI/DecimalO* and *LineI/DecimalO* respectively. In this case, each format gives a decimal approximation to the exact fractional result.

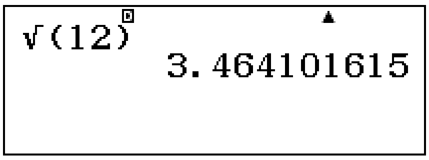




To illustrate further differences, the first two screens below are in Math input format, but the first shows the result as an exact number with Math output (*MathO*) and the second as an approximate number with Decimal output (*DecimalO*); the S⇔D key will allow you to switch between these two representations. The third screen is in *LineI/DecimalO* format; however, use of the S⇔D key will *not* convert the number to an exact representation.







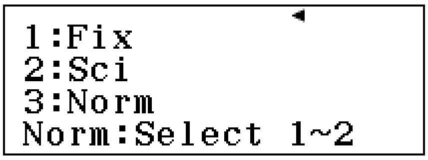
We suggest that you experiment with these formats to suit your purposes and to understand the differences between them, although we suggest that you almost always use Math input mode.

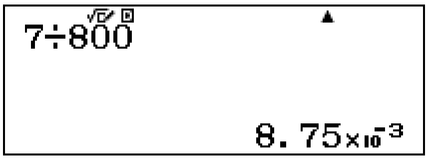
Angles
The calculator can accept angles in degrees, radians or gradians, and the choice made in SET UP is shown in the screen display with a small D, R or G. We will discuss the differences between these and their respective uses in Module 5. Your choices can always be over-ridden in practice for any particular calculation, however, which is also explained in Module 5. Most people leave their

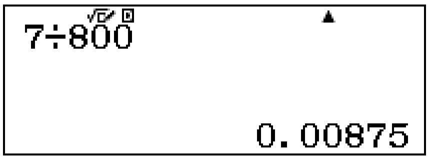
calculator in degrees if they are generally concerned with practical problems or radians if they are generally concerned with theoretical problems. We suggest that you leave it in degrees for now.

Number format
There are a few choices for the way that numbers are displayed as decimals. You can select *Fix* (1) to specify the same number of decimal places for all results, select Scientific notation, *Sci* (2) for all results or choose Normal notation, *Norm* (3) for all results. It can sometimes be a useful idea to choose *Fix* or *Sci* (e.g., to ensure that all results are given in similar ways, especially if all results are money values), but we think it is generally best to choose *Normal* decimal formats, allowing the calculator to display as many decimal places as are appropriate.

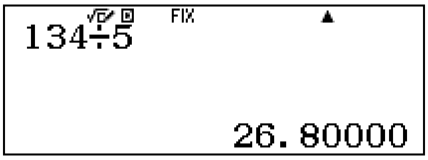
When *Normal* is chosen, there are two choices available, called *Norm 1* and *Norm 2*. These are almost the same, except that using *Norm 1* will result in scientific notation being used routinely for small numbers before *Norm 2* will do so. For example, the second two screens below show the same calculation as a decimal after selecting *Norm 1* and *Norm 2* respectively, and using SHIFT = to force a decimal result.

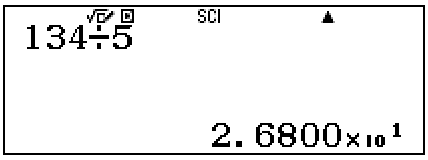


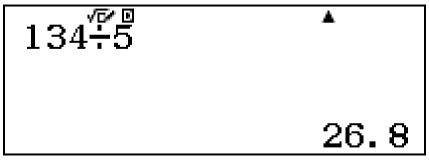




We suggest that it is generally better to choose *Norm 2*, but you should decide this for yourself, as it is mostly a matter of personal preference and also depends on the kinds of calculations you generally wish to complete. Here is the same number (134÷5) represented as decimals in the three formats *Fix*, *Sci* and *Norm* respectively:

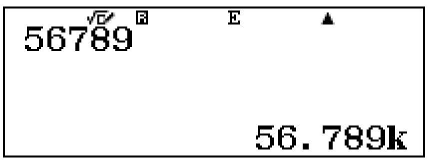


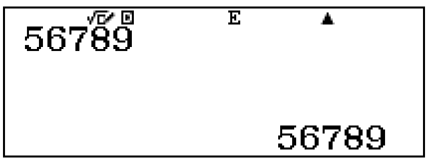


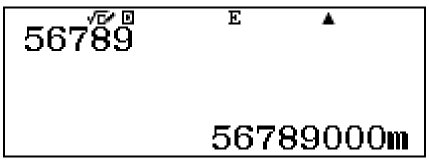


Notice that both *Fix* and *Sci* are shown in the display when they have been set up as the chosen format. When you select *Fix*, you need also to select the number of decimal places to be used (five are shown in the first screen above). When you select *Sci*, you need also to select the number of digits to be displayed (five are shown in the screen above, so there are four decimal places showing and one digit to the left of the decimal point).

Engineering symbol
As noted earlier in this module, engineering notation can be used sometimes to help interpretation. One way of doing this is to insert engineering symbols, instead of relying on powers of ten. The first SET UP screen allows you to choose whether or not to use these symbols. When they are used, as shown below, standard abbreviations for powers of ten are used. The screens below show a distance of 56 789 m can be interpreted as 56.789 km or as 56 789 000 mm, by converting numbers through successively tapping the ENG key.



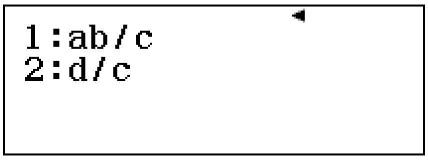




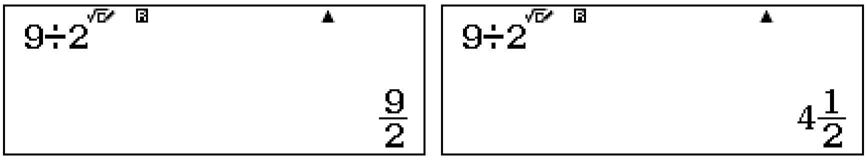
In these cases, notice that the use of Engineering symbols is shown by the small E at the top of the screen.

Fraction Result

The second SET UP screen shown below offers a choice of two ways of giving Fraction Results: as mixed fractions (using [1] ab/c) or proper fractions (using [2] d/c).



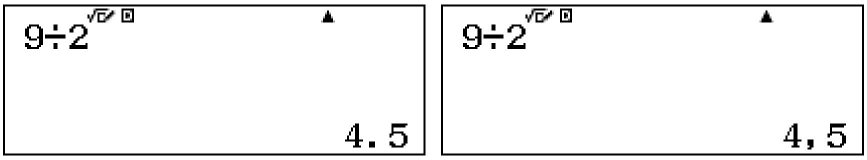
As shown in the next module, results can readily be converted with ($a\frac{b}{c} \leftrightarrow \frac{d}{c}$) from one of these to the other (via [SHIFT] [S⇔D]), so the decision is not very important.



To illustrate the effects of the choices, the same calculation has been completed in each of these two formats above.

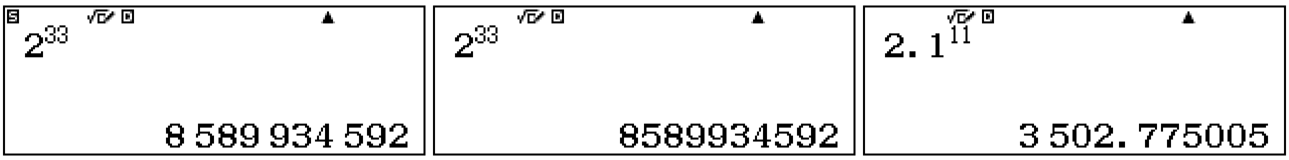
Decimal Mark

In the third SET UP screen you can select [3] *Decimal Mark* to choose between a dot or a comma for a decimal point in the calculator display. Make the choice that is appropriate for your country. Here are the two choices:



Digit Separator

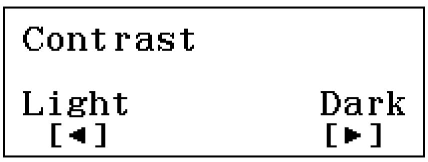
Another choice in SET UP concerns the display of large numbers. If [4] *Digit Separator* is chosen, then large numbers are separated by a space every three digits, rather than being written with no spaces. Many people find it easier to read numbers in that form, and we will often use it in these modules. The first two screens below show examples with and without this feature chosen. The third screen shows how digit separation works for a large number that is not an integer; notice that only digits to the *left* of the decimal point are grouped.



ClassWiz uses a space instead of a comma or a dot to separate digits, to avoid any confusion with various national number systems, which use these two symbols for decimal marks.

Contrast

You can select *Contrast* to adjust the contrast of the screen to suit your lighting conditions.

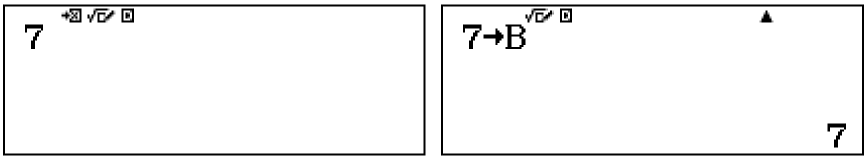


Hold down, or tap repeatedly, the ◀ or ▶ cursor keys until the contrast is suitable. It is generally best to be somewhere between too light and too dark, but this is also a personal preference.

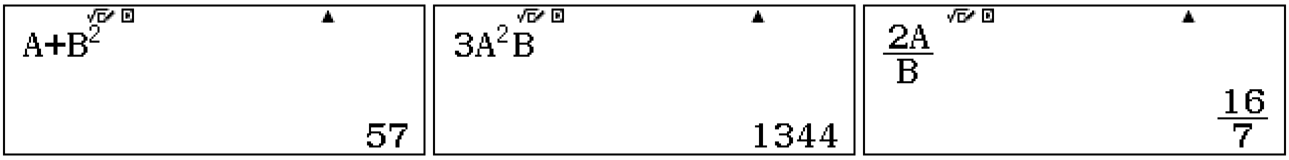
Memories

Calculator results can be stored in memories and retrieved later. This is convenient for recording values that you wish to use several times or for intermediate results. Both variable memories (labelled A to F as well as X and Y) and an independent memory (labelled M) are available.

To store a result that is already showing on the calculator into a variable memory, tap STO. This applies to a number you have just entered or to the result of a calculation just completed. Notice that the calculator display then shows a symbol to indicate a value being stored in a memory. Finally, tap the memory key for the variable concerned, shown with red letters above the keys on the keyboard. For example, the memory key for B is °''' and that for *x* is). Notice that there is also a special key for *x* at the top right of the calculator; this uses the same memory as *x*. The screens below show the process of storing a value of 7 into memory B. Notice that neither the ALPHA key nor the = key is used here.

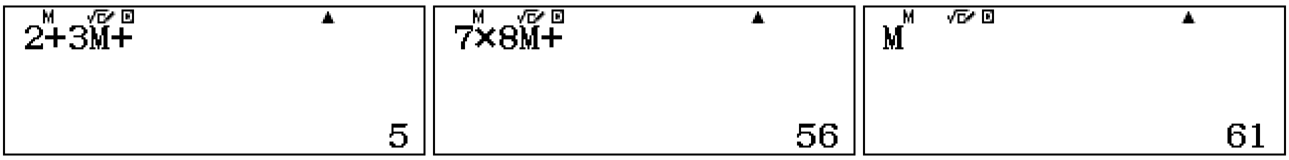


You can now regard B as a variable, with a present value of 7. To recall the present value of a variable, tap the ALPHA key, followed by the variable key. It is not necessary to use the ALPHA key if the variable *x* is used: merely use the *x* key. Variables are used on the calculator in the same way that they are in algebra, as shown below, after storing a value of 8 to memory A.

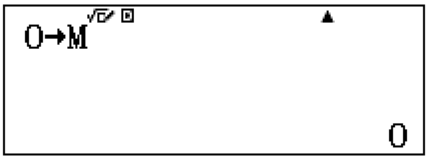


To change the value of a memory variable, you need to store a different number into the memory, as storing *replaces* any existing value. Turning off the calculator or changing modes will not delete the memory contents. You can clear the memories with SHIFT 9, but it is not necessary to do so, since storing a number replaces the existing number.

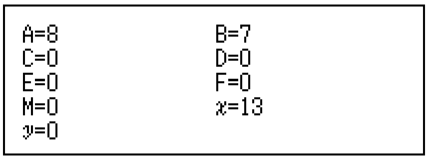
The Independent memory (M) works a little differently from the variable memories, although you can use it as a variable memory if you wish. The difference is that you can easily add or subtract numbers to or from the memory, using M+ or SHIFT M+ (M-). The first two screens below show M being used to store (2 + 3) + (7 x 8), while the third screen shows the result being recalled.



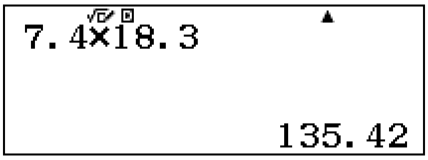
It is not necessary to tap the = key at any stage here except when recalling the value of M. Notice that whenever M contains a non-zero number, the screen display shows an M to alert you to this. The easiest way to delete the contents of M is to store 0 in the memory (with 0 STO M+), as shown below. You should do this before starting a new series of additions to M.

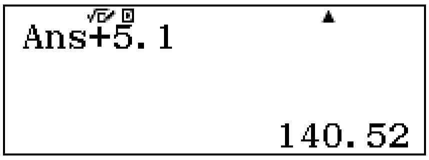


If you are using memories regularly, it can be difficult to remember which values are stored in them. A useful RECALL command, [SHIFT] [STO], recalls all nine calculator memories on the screen at once, as shown below.

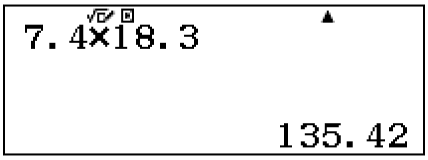


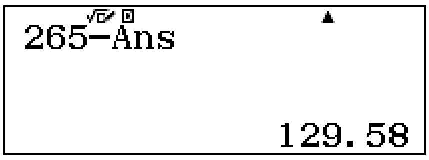
A very useful calculator memory is [Ans], which recalls the result of the most recent calculation. You might have seen this appearing when doing a succession of calculations. For example, the first screen below shows the calculator being used to find 7.4 x 18.3. When [+] [5] [•] [1] [=] is then pressed, the calculator assumes that the value of 5.1 is to be added to the previous result, which it refers to as *Ans*, since there is no number before the + sign. (*Ans* was not entered by the user.)





When a previous result is not to be used immediately, unlike the above case, then the *Ans* memory can be recalled with [Ans], as shown below to find 265 – (7.4 x 18.3) after first calculating the value in parentheses:

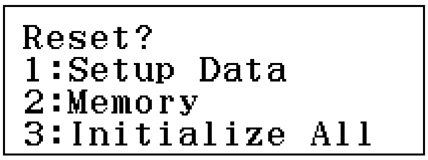


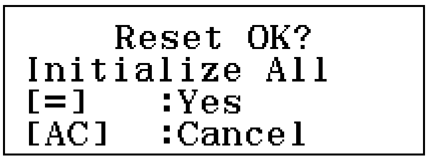


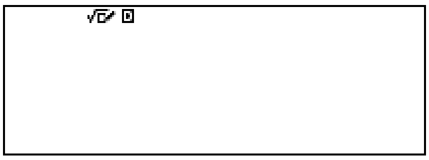
We will use the [Ans] key extensively in Module 14, where it is especially useful.

Initializing the calculator

Finally, while it is not necessary to initialize the calculator before use, this is the easiest and most efficient way to restore all of the calculator's default settings at once. After turning the calculator on with the [ON] key, tap [SHIFT] and [9] to show the RESET menu, shown in the first screen below.







Tap [3] to select *Initialize All* and then tap the [=] key to complete the process. The middle screen above shows the resulting message, while the third screen shows that the default settings involve Math input format and Degrees for angle measures, and that all the previous commands have been lost, as well as any data stored in the calculator.

As the screen above shows, you can choose to clear only the SET UP or the memories, if you wish.

Many people never need to reset their calculator, so you should not feel that you should do so.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. Use the calculator to find 73 + 74 + 75 + 760 + 77 + 78. You should get a result of 1137. Then edit the previous command, changing the 760 to 76, and check that the resulting sum is now 453.

2. Express the square root of 32 as an exact number and as a decimal number.

3. Find cos 52^o.

4. The hypotenuse of a right triangle with shorter sides 7 and 11 can be found by calculating $\sqrt{7^2+11^2}$. Give this length as a decimal.

5. Use the calculator to evaluate $\frac{22}{3}\div\frac{14}{17}$. Then express the result as a mixed fraction.

6. Find 10 more than the seventh power of 17.

7. Find $\log_3 81$.

8. When each person in a room of *n* people shakes hands with each other person in the room, there are nC_2 handshakes. How many handshakes will there be if there are 38 students in a room and one teacher?

9. Evaluate 38!, which is the number of different orders in which the students in the previous question could line up outside their classroom.

10. Find the absolute value of 3.4 – 7.81, which is represented in standard mathematical notation as $\left|3.4-7.81\right|$.

11. Use the calculator to evaluate √2 + √3 correct to three decimal places.

12. Evaluate $\left(\sqrt{4.1^2+5.3^2}\right)^5$.

13. As noted in the module, the population of the Philippines was 92 337 852 in the 2010 census. If the population keeps growing at 2%, use the calculator to find out approximately when the population will reach 150 million. (Hint: To do this, enter a command and edit it successively until you get the desired result.)

14. Change the Mode of the calculator to use natural display but to give answers always as decimals in Line mode. Check that you have done this successfully by evaluating √50.

15. Change the calculator SET UP to give all results with two decimal places. Check that this has been successful by finding a decimal value for the square root of 11.

16. Find the square root of 1.4 x 10^{17}, using the $\boxed{\times10^x}$ key to enter the number.

17. Assign memory variables A, B and C the values of 7, 8 and 9 respectively. Then evaluate AB^2C. Change the value of C to 5 and re-evaluate the expression.

18. Calculate the square of 34.5, but do not write down the result. Use the *Ans* memory to divide 8888 by your result.

Notes for teachers

This module is important for new users of the *ClassWiz*, as it deals with many aspects of calculator use that are assumed (and so are not repeated) in other modules. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of calculations.

Depending on the age and sophistication of the students, some parts of the introduction can safely be overlooked for later.

As a general principle, encourage students to look carefully at their calculator screens and to make sure that they understand what they are seeing. One valuable strategy is to ask students to predict what will happen before they tap the [=] key to complete a calculation. Making a prediction will help them to consider carefully what the calculator is being asked to do, and give them a stake (however small) in seeing that the answer produced is what they expected. Indeed, it can be a powerful and helpful lesson for students to make predictions that turn out to be incorrect, since this may encourage them to consider why their prediction did not eventuate.

It is generally a good idea for students to work with a partner, especially when they get stuck, so that they can discuss their ideas together and learn from each other.

If an emulator and data projector are available to you, you may find it helpful to demonstrate some calculator operations to the whole class, or allow students to do this. This is also a good opportunity to emphasise the need to understand exactly what is showing on the screen and to model the process of predicting the effects of a particular operation.

The Exercises at the conclusion of this Module are best completed by students individually to develop their expertise with using the calculator efficiently. We have provided brief answers to these exercises below for your convenience; some teachers may be comfortable giving students the answers along with the exercises, so that they can check their own progress and seek help when necessary. This is your choice, of course. You may find it convenient to demonstrate efficient methods for some of the exercises in this case, if an emulator and data projector are available.

Later Modules will also comprise some Activities for students to explore, but this introductory module is focussed on making sure that general calculator operations and settings are well understood, which will make later work with calculators more efficient. Nonetheless, some students may find aspects of mathematics to explore as a result of their introduction to calculator capabilities. We suggest that it is a good idea to allow them to do this, either by themselves or with other students, as many mathematical learning opportunities are offered by engaging with a classroom tool of this kind.

Answers to Exercises

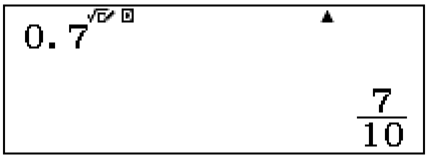
1. self-checking 2. $4\sqrt{2}$, 5.657 3. 0.616 4. 13.038 5. $\frac{187}{21}$, $8\frac{19}{21}$ 6. $17^7 + 10 = 410\ 338\ 683$
7. 4 8. ${}_{39}C_2 = 741$ 9. 5.230×10^{44} 10. Use [SHIFT] [(] to get $|3.4 - 7.81| = 4.41$ 11. Use [SHIFT] [=] to get 3.146 directly 12. 13 508.7714 13. Use $92337852 \times 1.02^{20}$ and edit the exponent to get the best approximation: about 25 years. So 2035 is the required year. 14. 7.071 (instead of $5\sqrt{2}$) 15. 3.32 16. 374 165 738.7 17. 4032; after storing C = 5, use [▲] to highlight the expression and tap [=] to get the new result of 2240 18. After finding the square of the number, use 8888 ÷ *Ans* to get 7.467

Module 2
Representing numbers

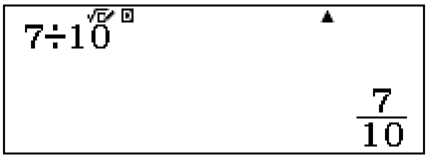
A number can be represented in many different ways. You can use your *ClassWiz* to show these and see how they are related to each other. In this module, we will use Calculation mode; tap MENU 1 key to do this. Make sure your calculator is set into Math mode for both input and output and Norm 2. Use SET UP to do this, if necessary.

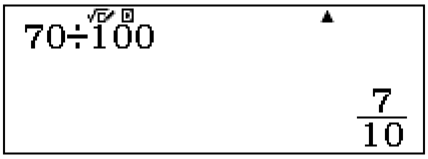
Representing decimals

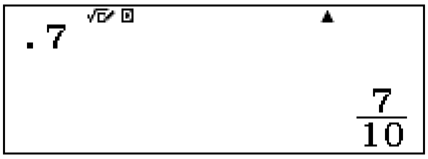
When you enter a decimal number into the calculator and tap the = key, the number will usually be shown in the form of a fraction. The calculator will usually choose the simplest fraction that it can, as the screens below show.



As you can see, 0.7 and 0.70000 are the same number, and each can be represented by the fraction seven tenths. There are other ways to represent this same number, as shown below.

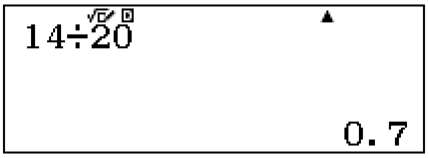






Although it is not generally advisable, people sometimes write decimal numbers between 0 and 1 without the initial zero, as above. The calculator will still recognise it as the same number, however.

If you prefer a number to be represented as a decimal instead of a fraction, you can tap the standard to decimal key S⇔D as soon as the standard version, in this case a fraction, appears. Notice that tapping the S⇔D key again will change the number back to a decimal. The two screens below show this process in place: you can toggle between a decimal and a fraction representation of 14÷20 by tapping the S⇔D key repeatedly. Notice that this is yet another way of representing the same number.

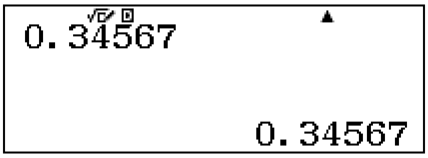


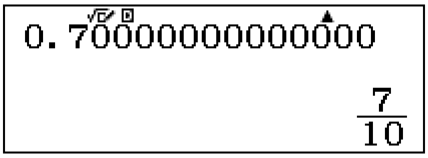
If you know in advance that you would prefer a number to be represented as a decimal, rather than a fraction (for example, at the conclusion of a calculation), then it is possible to do this directly without producing the fractional form first: enter the number and tap the SHIFT key before tapping the = key. (Notice the approximation symbol (≈) above the = key) Try this for yourself.

The calculator will not *always* represent a decimal as a fraction. If the denominator of the simplest fraction requires more than four digits, it will automatically represent the number as a decimal. For example, the decimal number 0.34567 can be represented as a fraction, with a large denominator:

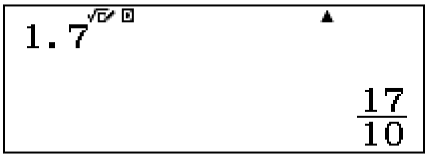
$$0.34567 = \frac{34567}{100000}$$

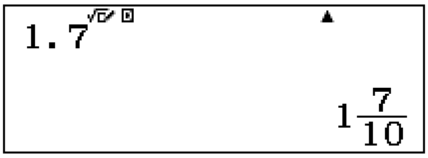
On the calculator, however, it will enter as a decimal number and the S⇔D key will not change it to a fraction, as shown below. Notice also below that a number with many decimal places will still be shown as a fraction, if the denominator needed is small enough.





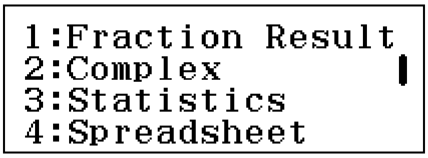
When a decimal number is larger than one, the fractions that represent it will have a larger numerator than denominator. Fractions of that kind can be represented in standard form in two different ways: as proper fractions or as improper fractions. The two possibilities for representing 1.7 are shown in the screens below:

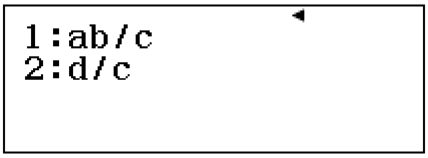




Each of these is a correct way to represent 1.7 as a fraction. The first shows an improper fraction, with numerator larger than the denominator and the second shows the fraction as a mixed number. The version that is displayed can be changed to the other version by tapping the SHIFT key and then the S⇔D key to toggle between the two using the ($a\frac{b}{c} \Leftrightarrow \frac{d}{c}$) command. If you tap SHIFT S⇔D repeatedly, you can switch from one form to the other, and choose which one you prefer.

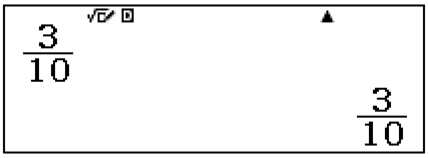
To change which version is used automatically by the calculator, use SET UP and tap the ▼ cursor to get the second screen. Tap 1 to select *Fraction Result*. Notice that you can use 1 or 2 to choose between 1: ab/c to give proper fractions or 2: d/c to give improper fractions each time, as shown on the screen below. Repeat the calculations above to check the effects.

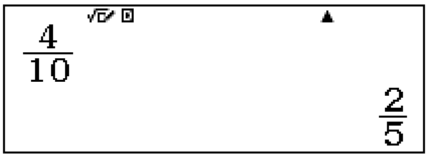




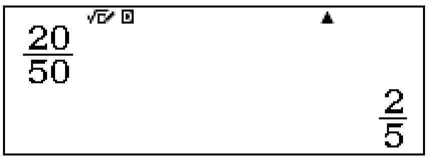
Representing fractions

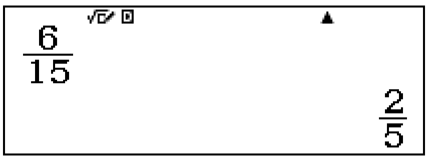
Fractions can be entered into the calculator using the fraction key and will be displayed as fractions when you tap the = key. If a fraction is already in its simplest form, it will be shown again when the = key is tapped. But if the fraction can be represented in a simpler form, then the calculator will do this automatically. The next two screens show examples of each of these two possibilities.

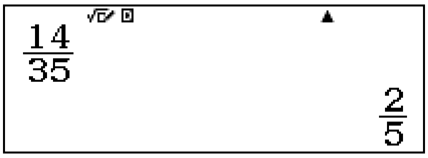


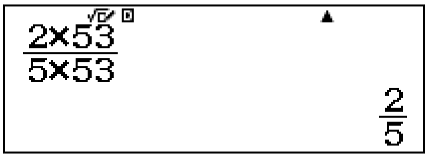


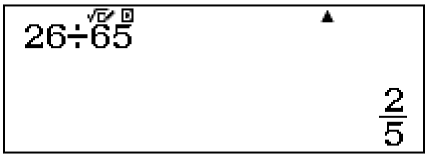
There are many different fractions that can be represented by a simplified fraction like two fifths. Here are five more examples:







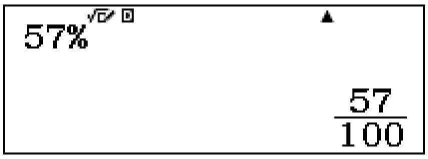


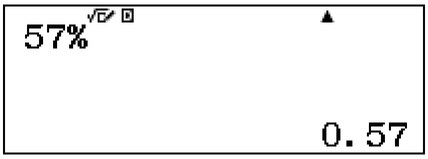


Look at these carefully. You can probably find many other examples of fractions that are represented by the calculator as two fifths. Because all of the fractions represent the same number (i.e. two fifths or 0.4), they are described as *equivalent* to each other. Equivalent fractions are very helpful for understanding how fractions work.

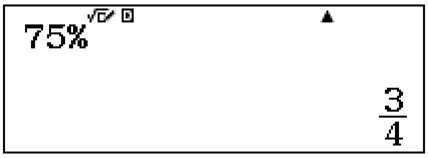
Representing percentages

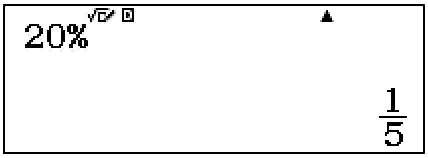
Percentages are special fractions, with denominator 100. So 57% is a short way of writing 57/100, which is the same number as 0.57. You can enter a percentage into the calculator using the (%) key, (obtained by tapping [SHIFT] and then [Ans].) Notice below that the standard representation of 57% is as a fraction, but you can see the decimal representation by using the [S⇔D] key.



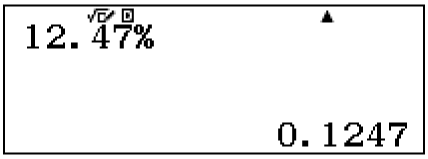


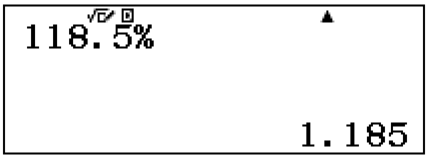
Some percentages can be represented as equivalent fractions with denominator less than 100, as the screens below show. The calculator does not automatically represent a number as a percentage.



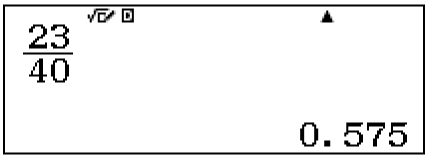


Notice below how percentages and decimals are related:





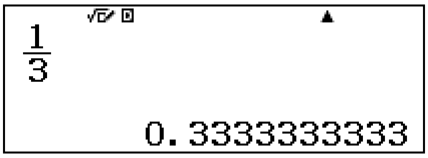
The number before the percentage sign is always 100 times the decimal representation (because percentages are fractions with denominator 100). So to represent a number as a percentage, first represent it as a decimal and then multiply the decimal by 100. You should be able to do the last step in your head.

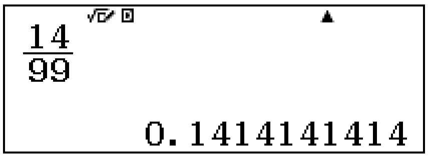


In the example above, to see what percentage 23 is out of 40, the calculator shows that 23/40 is 0.575 so you should be able to see that this is the same as 57.5%.

Recurring decimals

You may have already noticed that many fractions, when represented as decimals, need the full calculator display to represent them. This is often because they are recurring (or repeating) decimals that have an infinite (never-ending) representation as decimals. Here are two examples:



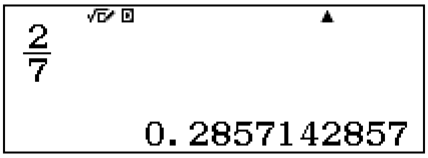


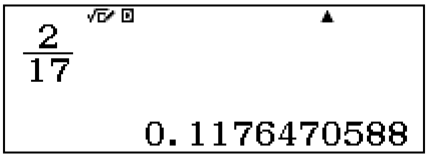
Calculators (and computers) always have a finite amount of space to represent numbers, so cannot represent infinite decimals completely. In these two cases, it is fairly easy to see the repeating patterns involved and to see how they will continue. In mathematical terminology, recurring decimals are often represented by showing just the repeating digits and a bar (or dots) over them. In this case, the two numbers above are commonly shown as

$$\frac{1}{3} = 0.\dot{3} \text{ and } \frac{14}{99} = 0.\overline{14}$$

The calculator does not add the recurring symbols: you need to do this yourself.

Often, the calculator needs to be interpreted more carefully than in the two examples above to see the pattern in a recurring decimal, and you will need to do some careful mathematical thinking or even some exploration. Here are two more difficult examples.

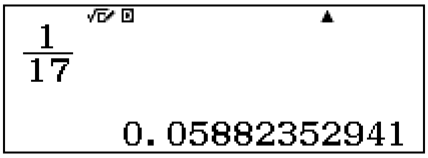


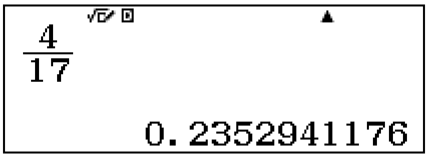


The first of these is easier to interpret than the second one, as the display has some repeating digits (2857), which suggests that the decimal representation is

$$\frac{2}{7} = 0.285714285714285714285714285714... = 0.\overline{285714}$$

In the second case, however, none of the digits displayed are repeating. To see the pattern involved, examine some other fractions with the same denominator of 17, as the next two screens show.



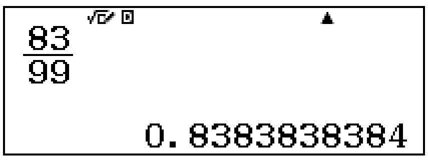


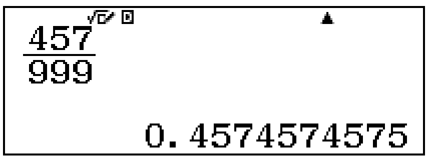
If you look at the last three screens carefully, you should be able to see the pattern of the recurring digits, which is too large to be shown on a single calculator screen.

$$\frac{2}{17} = 0.11764705882352941176470588235294 1176... = 0.\overline{1176470588235294}$$

In this case, a little experimentation has shown the pattern, which can be found because all the fractions with a denominator of 17 have the same recurring digits. (Not all fractions behave in this way … you will need to experiment further with some others.)

Sometimes, recurring decimals will be rounded to fit the screen, which requires you to look carefully at what you see. Here are two examples, for which the final digit has been rounded up, but for which you need to understand how the calculator works to interpret the screen correctly.





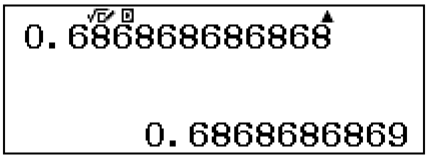
In this case, the correct representations are

$$\frac{83}{99} = 0.838383838383838383... = 0.\overline{83} \text{ and } \frac{457}{999} = 0.457457457457457457... = 0.\overline{457}$$

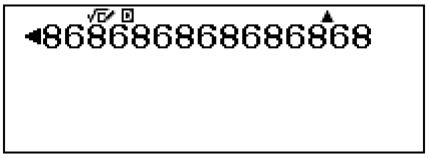
You also need to think carefully about rounding when converting recurring decimals to fractions. The calculator displays ten places of decimals, but will allow you to enter more places, which is sometimes necessary. For example, if you enter the decimal 0.686868686868 with twelve places of decimals and tap =, the calculator interprets the number as the decimal

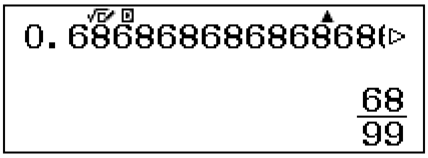
$$\frac{686868686868}{1000000000000} = 0.686868686868$$

and so rounds it to ten decimal places to display 0.6868686869, as below. This is quite appropriate, as the decimal entered is not in fact a recurring decimal. (It terminates after 12 decimal places.)



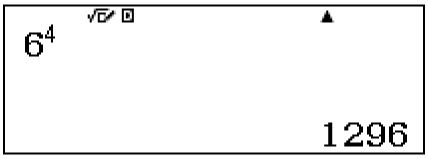
However, you can enter more places of the recurring decimal, even though they do not all fit on the calculator display, as the first screen below shows: notice the arrow on the left of the number. When you tap =, the calculator will now *interpret* the number entered as the recurring decimal for 68/99, which is $0.\overline{68}$. Notice the arrows on the right of the number in the second screen below. Experiment with this idea for yourself.



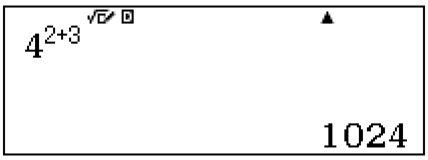


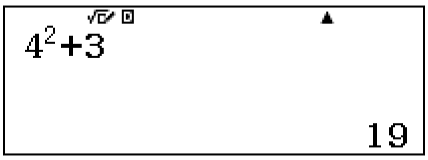
Powers

Powers are involved when a number is multiplied by itself repeatedly. So, 6 x 6 x 6 x 6 is called 'six to the power four' or just 'six to the fourth' and is written as 6^4. There is a special powering key $x^{\blacksquare}$ on the calculator to evaluate powers. To use this, first enter the base (in this case 6), tap $x^{\blacksquare}$ and then enter the power (sometimes called an exponent) of 4. Notice that the power is written in a slightly smaller font than the base. Tap = to see the result, as shown below.



If an exponent is more than just a single number, you can enter it directly on the calculator. For example, look carefully at the first screen below, which shows 4 raised to the power of 2 + 3.

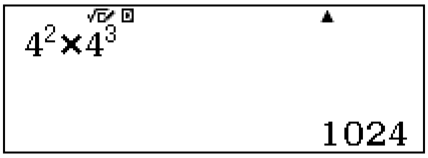


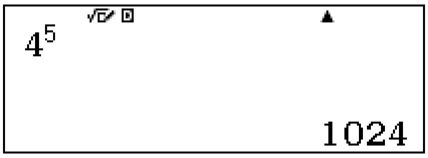


If you wish to do calculations involving exponents, you may first need to move the calculator cursor out of the exponent, using the ▶ key, in order to complete writing the power and continue writing an expression. Notice above that there is an important difference between 4^{2+3} and $4^2 + 3$.

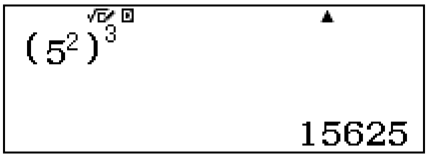
The first of these expressions means 4 x 4 x 4 x 4 x 4, while the second means 4 x 4 + 3, which of course is quite different.

It is an important mathematical relationship that when powers to the same base are multiplied, their exponents can be added. For example 4^2 x 4^3 = (4 x 4) x (4 x 4 x 4) = 4 x 4 x 4 x 4 x 4 = 4^5 = 1024. Look carefully at the screens below to see these represented on the calculator. You will need to use the ▶ key before entering the multiplication sign.



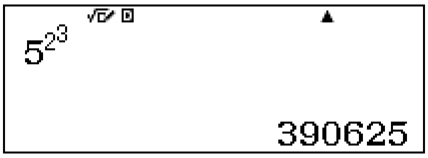


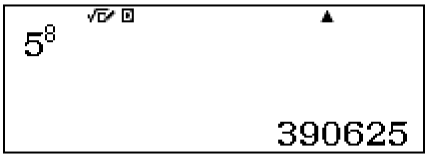
Powers themselves can be raised to powers on the calculator, but you need to be careful to enter the expressions correctly. For example, the third power of 5^2 is 5^2 x 5^2 x 5^2, which can be written as $(5^2)^3$. You will need to enter the parentheses on the calculator and also tap the ▶ key after entering the first power before closing the parentheses, as shown below.



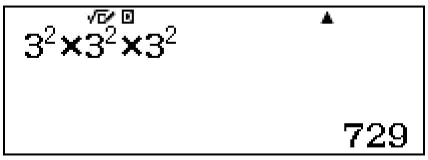
Check for yourself that the result is the same as 5^6, since the powers can be added, as noted earlier: 2 + 2 + 2 = 6. Notice also that the two powers can be multiplied in this case: 2 + 2 + 2 = 3 x 2 = 6. Check for yourself also on the calculator that $(5^2)^3$ gives the same result as $(5^3)^2$, since 3 x 2 = 2 x 3.

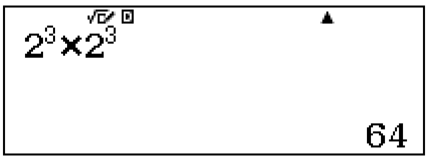
If the parentheses are omitted, the calculator assumes that a power applies to the number immediately before it, so the expression below, obtained by using the $x^{\blacksquare}$ key twice in succession, is interpreted as 5 to the power 2^3 or 5 to the power of 8.



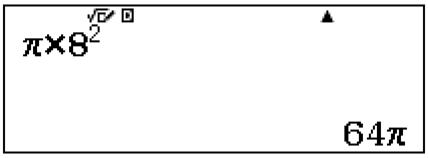


Although you can always use the $x^{\blacksquare}$ key for evaluating powers, sometimes it is convenient to use the special keys x^2 and x^3 to find the second and third powers (called the *square* and the *cube*, respectively). Notice that the cube key requires you to tap SHIFT and then x^2. These commands generally give the same results as using the power key (although they are calculated slightly differently by the calculator), but are useful because it is unnecessary to use the ▶ key to evaluate complicated expressions. For example, in each of the two screens below, it was unnecessary to use the ▶ key.

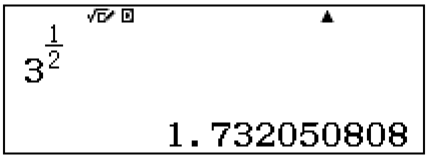


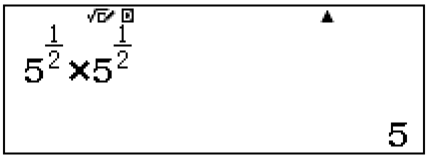


These commands are helpful for evaluating mathematical expressions involving squares and cubes, for example to find areas and volumes. The screen below shows the calculator being used to find the exact area of a circle of radius 8 cm.



You may be surprised to find that you can use exponents that are not whole numbers, such as fractions and decimals. A good example is the power of a half, which is shown below, using the fraction key after the power key in each screen. In the second screen, be careful to use the key twice to move the calculator cursor out of the first fraction and then out of the exponent, before tapping the multiplication sign.





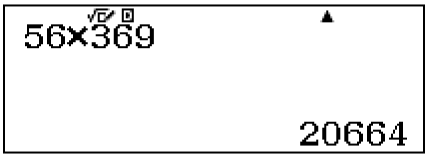
Notice especially that the second screen shows again that powers to the same base can be added, in this case giving the result that $5^{½} \times 5^{½} = 5^{½+½} = 5^{1} = 5$.

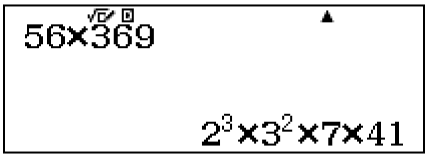
Factors

The *factors* of a (whole) number are the (whole) numbers that divide evenly into the number. For example, the factors of 12 are 1, 2, 3, 4, 6 and 12. Most whole numbers have several factors (and are called *composite* numbers), while less frequently, some numbers have only two factors: the number itself and 1. These are called *prime* numbers, and are very important in mathematics, especially in the branch of mathematics called Number Theory.

The first few prime numbers are 2, 3, 5, 7, 11, 13, …

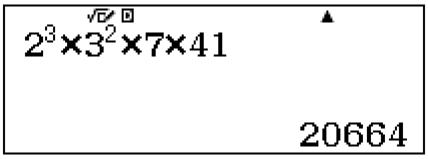
The calculator has a special command, FACT, to represent whole numbers as a product of their prime factors. For example, the first screen below shows a product of two whole numbers, represented as another whole number, 20 664). After the result is obtained, tap SHIFT ° ' " to use the FACT command to obtain the prime factors of the number, as shown in the second screen.



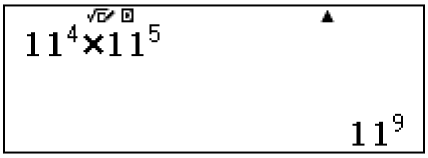


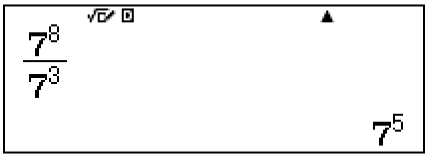
Since 1 is not a prime number, it is not included as one of the prime factors. A positive integer can be represented as a product of only one set of prime factors (described in mathematics as *unique factorisation*), which would not be the case if 1 was regarded as a prime number.

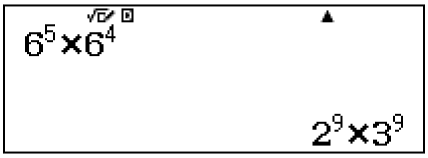
To verify this factorisation, you can use the calculator to multiply the set of prime factors together:



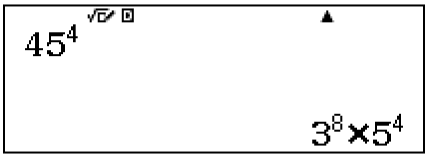
You can use this command to explore how powers of whole numbers are combined when multiplied or divided, as shown below:

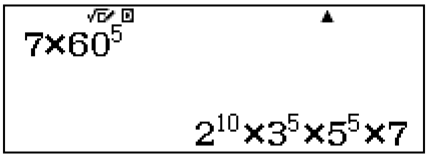




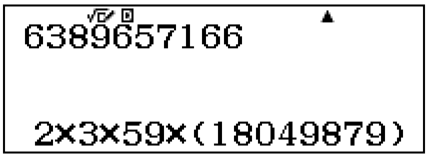


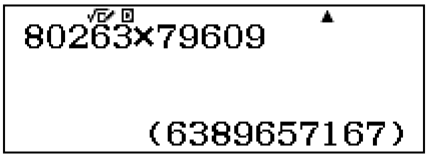
Notice the difference between the first and third examples above: the calculator does not represent the product of 6^5 x 6^4 as 6^9, since 6 is not a prime number. Instead, it represents the result in terms of its prime factors of 2 and 3. The next two screens also show how the calculator represents numbers as a product of their prime factors. Study them carefully to see how.

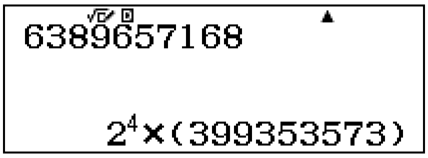




It is often difficult to determine the factors of integers (which is used by ATMs and other devices to guarantee Internet security), and the calculator will not be able to do so for some large numbers. There are three examples of large numbers below, after the FACT command has been used.

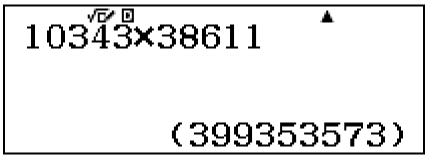






In the first example, the calculator is unable to find all the prime factors of 6 389 657 166, as it cannot determine whether or not 18 049 879 is prime. This is shown with the parentheses around the result. In the second screen, the calculator is unable to determine any prime factors of the number 6 389 657 167, although it is clearly the product of two large prime numbers (80 263 and 79 609). Similarly, the calculator has identified in the third screen that 6 389 657 168 has four prime factors of 2, but is unable to determine whether or not there are any prime factors of the large number, 399 353 573, again shown by the use of parentheses.

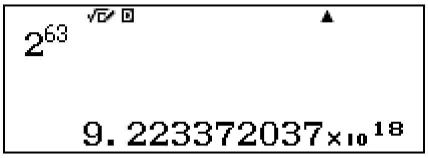
It is important to interpret these displays correctly: numbers in parentheses might be prime numbers, but also might be composite numbers; the calculator is unable to determine which is true. In this case, the number in parentheses in the first screen *is* a prime number, while that in the third screen is *not* a prime number, as shown below, but the calculator is unable to find its factors.

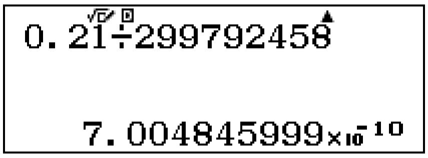


Similarly, if numbers are so large that they can only be represented on the calculator using scientific notation, then it will not be possible for the *ClassWiz* to find their prime factors.

Scientific notation

As noted in Module 1, you can use SET UP to display numbers in scientific notation. However, when numbers are too large or too small to be displayed as decimals or fractions, the calculator will automatically represent them using scientific notation. For example, there is a famous fable about the inventor of chess, who requested a reward of a grain of rice on the first square, twice as much on the second square, twice as much again on the third square, and so on for all 64 squares. So the number of grains of rice on the 64^{th} square is shown below.





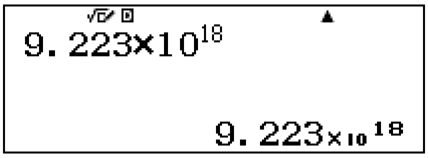
Similarly, given that light is assumed to travel at 299 792 458 metres per second, the number of seconds taken for light to travel across an A4 page 210 mm wide is shown above. In each case, the calculator has represented the result in scientific notation.

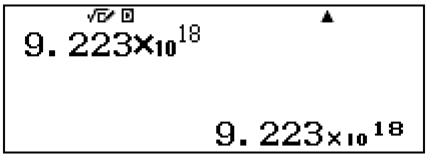
Both answers are approximations, since the calculator has only a limited number of digits that can be shown. Thus, the chessboard result is 9.223372037 x 1 000 000 000 000 000 000, or

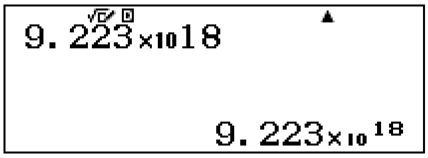
$$9.223372037 \times 10^{18} \approx 9\ 223\ 372\ 037\ 000\ 000\ 000$$

In fact the correct result (too large for the calculator to show) is 9 223 372 036 854 775 808, which has been rounded by the calculator.

Numbers can be entered in the calculator directly in scientific notation, using the multiplication key and the power key or the special power of ten key. However, it is much more efficient to use the special scientific notation key [×10^x] instead. These three methods of are shown below.

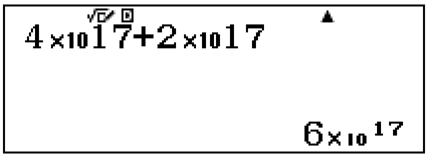


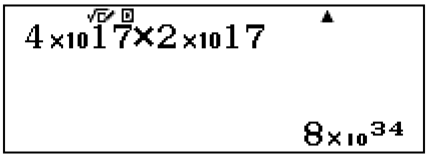


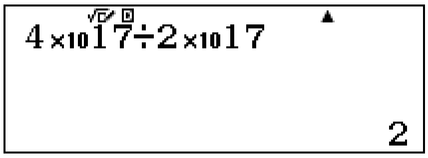


After entering 9.223, the first screen on the left shows [×] [1] [0] [x■] [1] [8] [=], the middle screen shows [×] [SHIFT] [log] [1] [8] [=], while the screen on the right shows [×10^x] [1] [8] [=]. The results are the same, but the number of keystrokes needed (7, 5 and 4) differs in each case. The special scientific notation key does not require you to use a separate key for either a multiplication sign or a power, which is why it is more efficient; notice that the screen display does not display the power of ten as a power, although the result indicates that it interprets it correctly as a power.

Entering numbers in scientific notation using the [×10^x] key helps you to see how the powers of ten are affected by computations. Study the screens below carefully to see examples of this for addition, multiplication and division of 4×10^{17} and 2×10^{17}.

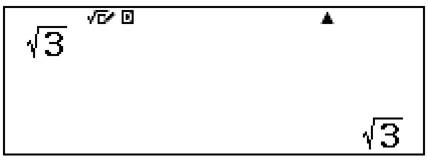


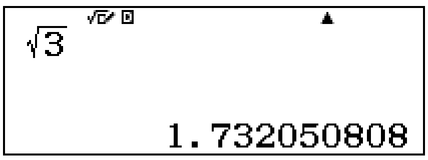




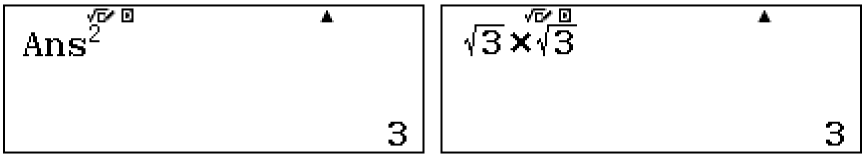
Roots

A *root* of a number is a number that can be raised to a power to produce the number. A *square root* of a number, for example, can be *squared* to produce the same number. On the calculator, the square root key [√■] is used to find square roots. Notice in the first screen below that an exact square root of 3 is given as a standard result, or you can tap the [S⇔D] key to get a numerical approximation. (Alternatively, you can use [SHIFT] [=] to get a numerical approximation directly.)



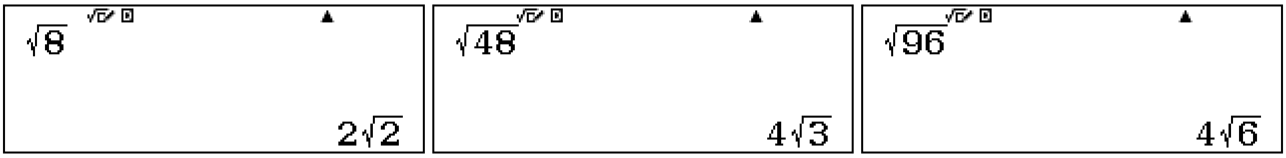


In either case, you can check that the square root is correct by immediately tapping the x^2 key above or, alternatively, multiplying the square root by itself. These two alternatives are shown below. (You need to use the ▶ key to exit from the radical sign √ in the second example.)

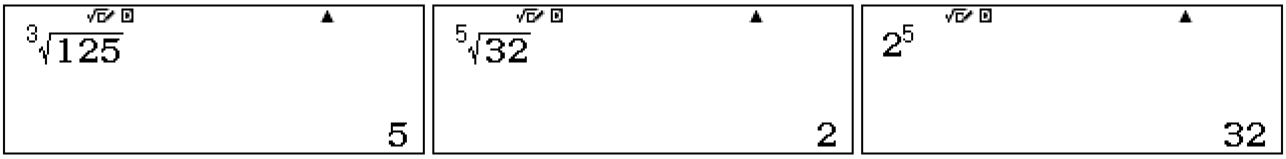


Notice that the calculator squares the last answer (called *Ans*) when you tap the x^2 key. The result is 3, since the square root of 3, when squared, must give 3, by definition. You may also have noticed that the square root of 3 was obtained in the previous section by finding $3^{½}$. Check by comparing the answers.

You can see some properties of square roots by finding square roots of composite numbers:

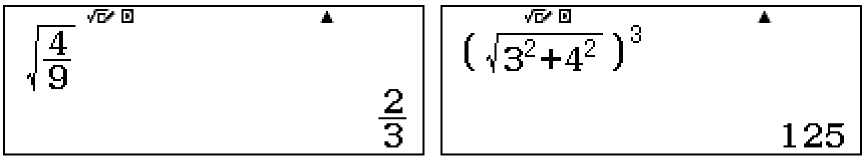


The calculator also has a direct command for other roots as well as square roots. Cube roots are available directly with a command (³√■) available through SHIFT √■, while other roots are available with the (■√□) key, accessed with SHIFT $x^■$. These two commands are shown in the first two screens below; notice the relationship between roots and powers demonstrated in the final two screens.



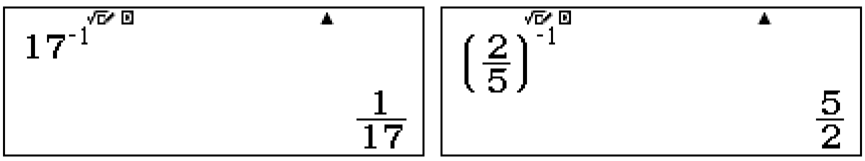
Unlike the commands for square roots, the calculator will provide only numerical approximations for other roots, unless they are whole numbers, as shown above.

You will find that calculations involving roots, powers and fractions can all be accomplished on the calculator, although you will need to use cursor keys to move around and sometimes parentheses to construct expressions in correct mathematical notation. Here are two more complicated examples:

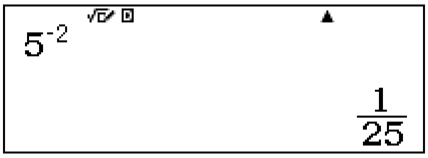


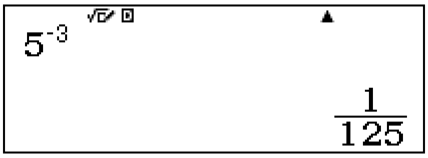
Reciprocals

A special key x^{-1} on the calculator is used to construct multiplicative inverses, or reciprocals, of numbers. The two examples below show the effects of using this key:



It is also possible to use the $x^■$ key with a negative exponent, using (−), instead of the x^{-1} key. Using the $x^■$ key is the easiest way to obtain other negative powers of numbers, as shown below.



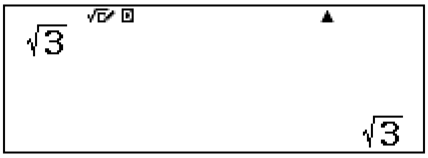


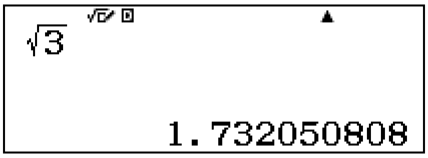
Rational and irrational numbers

Rational numbers are those that can be expressed as ratios of whole numbers. These are fractions, terminating decimals or whole numbers, such as 3, ¾, 16% or 3.725. Numbers that cannot be expressed in this way are *irrational* numbers. These are most radicals, logarithms and trigonometric ratios as well as special numbers like π. When represented by decimals, irrational numbers require an infinite (never-ending) number of decimal places, with no patterns in the digits.

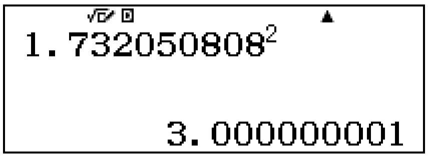
On the calculator, irrational numbers such as √3, $2^{1.2}$ and π can be represented exactly using standard mathematical notation or represented approximately, using decimals. It is important to realise that the decimal approximations give only the first few decimal places of an infinite number and are not exact. This is not merely a limitation of the calculator – the world's largest supercomputer is the same in this respect.

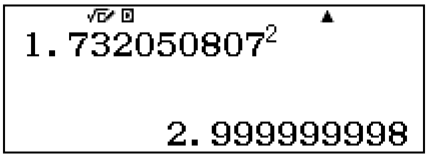
You can use your calculator to check that a numerical approximation to an irrational number shown on the calculator is merely an approximation. The screens below show both the exact value and the calculator's approximation for √3, as noted earlier in this module.





If you now enter the approximation as a decimal, and square the result, you will see that the calculator's approximation is too large:

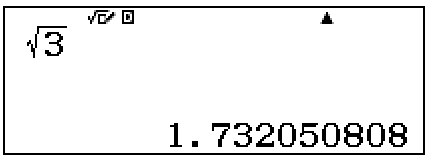


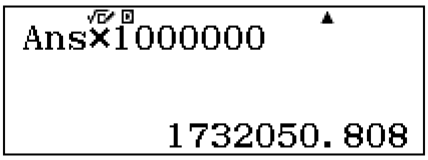


However, if the calculator had displayed the next smallest approximation to √3, with the same number of decimal places, the right screen above shows that it would be too small. So the approximation is the best available for the number of decimal places available. These two screens make it clear that 1.732050807 < √3 < 1.732050808. The calculator displays 1.732050808, indicating that the infinite decimal expansion is closer to 1.732050808 than to 1.732050807.

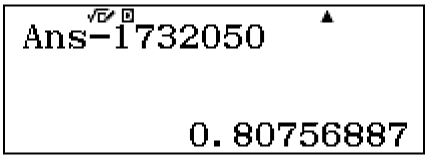
The calculator stores more digits internally than it displays, although it cannot store the infinite number of decimal places needed to represent the irrational number √3 *exactly*.

You can use a small trick to see the hidden decimal places stored in the calculator by first multiplying the value by a large power of ten and then by subtracting the integer part to leave only the remaining decimal places. For example, multiplying the approximate value by a million gives the same ten digits:



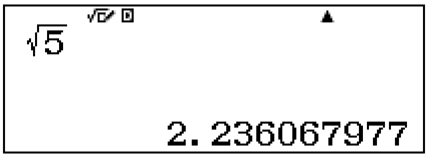


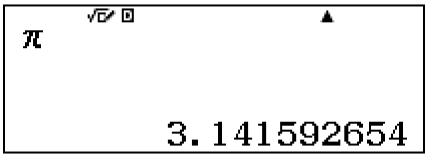
Now subtract the integer part, 1 732 050, to see further decimal places, which were previously hidden inside the calculator because the display is limited in size:



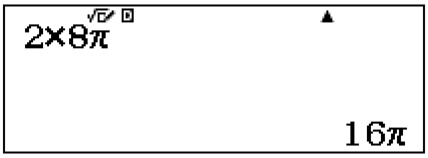
This shows that the *ClassWiz* works with the approximate value of √3 ≈ 1.73205080756887 internally, but displays only 1.732050808 on the screen. This is why it gave an apparently exact result previously when the $\boxed{x^2}$ key was used to square a numerical approximation. Of course, even the more accurate approximations to irrational numbers used internally are still only approximations, as the number of decimals required for an irrational number is infinite.

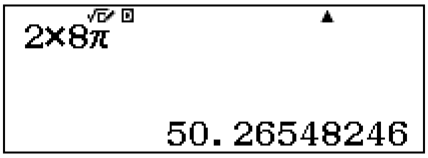
You can use a similar method to see for yourself that the calculator uses √5 ≈ 2.23606797749978, although it displays only √5 ≈ 2.236067977 and uses π ≈ 3.1415926535898, although it displays only π ≈ 3.141592654.



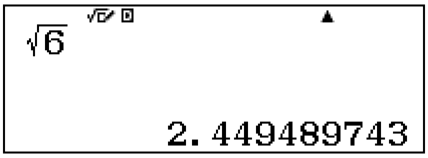


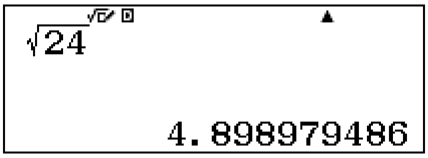
In practice, with everyday computations, only a small number of decimal places may be used to approximate irrational results. For example, we may choose to express the circumference of a circle of radius 8 cm as 50.3 cm, to the nearest 0.1 cm, instead of the exact, irrational, value of 16π cm.

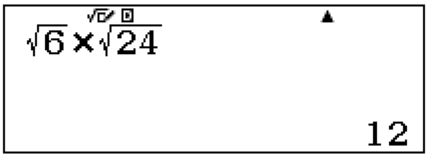




You may be surpised to realise that, although irrational numbers can be represented as decimals that require an infinite number of decimal places, some products and powers of irrational numbers are in fact rational. Here is an example:

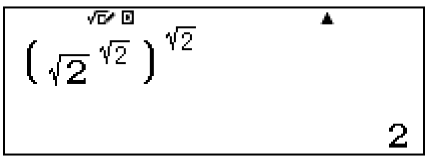






In the first two screens above, approximations to an infinite number of decimal places is shown, since both $\sqrt{6}$ and $\sqrt{24}$ are irrational. However, their product is $\sqrt{144}$ = 12 is an integer and so is a rational number. The first two screens show approximate results, while the third result is exact.

Similarly, the example below shows that irrational numbers raised to irrational powers can sometimes produce an integer, which is rational:



Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. Represent 0.45 as a fraction, simplified as much as possible.
2. Find $\frac{7}{9}+\frac{8}{11}$, expressing your answer as both a proper fraction and an improper fraction.
3. Express 43 as a percentage of 80.
4. Find 57% of 16.
5. A girl paid a deposit of 15% for a television set. If the deposit was $69, what was the full price of the TV set?
6. Find all the prime factors of 12 345 678.
7. Give the recurring decimal for 4/7.
8. What fraction is represented by the recurring decimal $0.\overline{126}$? Check your answer on the calculator by converting the fraction to a decimal.
9. Evaluate 7^6.
10. Evaluate $4^5 + 12$
11. Determine whether or not $2^{29} - 1$ is a prime number.
12. Evaluate $7^3 \times 7^3 \times 7^3$ and check that it is the same as 343^3.
13. Predict which is larger: $(6^2)^3$ or $(6^3)^2$? Use the calculator to check your prediction.
14. (a) Enter and evaluate the following three expressions on the calculator: $(5^2)^3$, $5^{(2^3)}$, 5^{2^3}.

    (b) Predict whether the calculator will interpret 2^{2^3} as 4^3 or 2^8. Use the calculator to check your prediction.
15. Evaluate $10^{½}$ and explain what happens when you find its square.
16. Find the fifth root of 1024. Check your answer by finding its fifth power.
17. Find the cube root of 23, correct to four decimal places
18. A square root of a whole number produced the result of $6\sqrt{2}$. What was the original number?
19. Find the square root of 7 correct to twelve decimal places.
20. In a forensic examination, a strand of human hair was reported to be 13 micrometres thick. (A micrometre is 10^{-6} metres.) Enter this thickness into the calculator to express it as a decimal.
21. Predict the value of 7^{-2}. Use your calculator to check your prediction.
22. Which of the following numbers, if any, are irrational?

    (a) $\sqrt{1521}$ (b) 13^{-3} (c) $\pi + 1$ (d) $\left(\sqrt[3]{17}\right)^3$ (e) 4.1425678423 (f) $\frac{357}{2143}$

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. If you invest \$500 at an interest rate of 5% per annum, then the amount of money you have (A) after t years is given by $A = 500(1.05)^t$.
   (a) After how many years will the value of the investment double?
   (b) What if you start with \$1000? How long will it take to double?
   (c) Choose an investment value for yourself, and investigate to see how long before it doubles.
   (d) How long does it take an investment to double if the interest rate were 7%? 8% …

2. Many fractions result in recurring decimals, which often show interesting patterns. For example, study the fractions with denominator of 7. Use the calculator to obtain 1/7, 2/7 and 3/7 and then predict the decimals for 4/7, 5/7 and 6/7, checking your predictions with the calculator.

   Examine other fractions with denominators that are prime numbers, such as 11ths, 13ths, 17ths and 19ths. Use the calculator will help you to get information in order to look for, and use, patterns.

3. Which of these numbers is largest? Which is smallest? Place them in order before using the calculator to check your decisions:

   $A = 3.5 \times 10^{12}$ $\quad$ $B = 6.9 \times 10^{11}$ $\quad$ $C = 4.2 \times 10^{12}$ $\quad$ $D = 2.0 \times 10^{13}$

   When asked to find $C + D$, Kim thought of D as 20×10^{12} and so got the sum as 24.2×10^{12}, which she then wrote in scientific notation as 2.42×10^{13}. Can you find quick and efficient ways like this of calculating with numbers in scientific notation? Check your methods by hand and then with the calculator. Compare your methods with those of other students.

   Use the methods you devised above to find $C \times D$, $A + C$, $B + C$ and $D - A$.

4. Ping Yee claimed that $x^{a+b} > x^a + b$, regardless of the values of x, a and b. Is he correct?

   Use your calculator to examine some examples of this relationship and look to find ways of explaining and justifying your conclusions to other students.

5. Is the following statement about negative powers true or is it false? $5^{-4} > 5^{-5}$

   Compare similar pairs of numbers, such as 6^{-4} and 6^{-5}, 5^{-8} and 5^{-9}, $6^{-1/2}$ and $6^{-1/3}$, 0.8^{-3} and 0.8^{-4}.

   Check your predictions with the calculator, but only after predicting which number is larger. Look for generalisations and explanations of them. Check your findings with other students.

6. You have seen that a number raised to the power of one half gives its square root, and a number raised to the power of one third gives its cube root. Check some examples of these on the calculator.

   What is the meaning of other fractional powers? Use you calculator to check that

   $32^{\frac{4}{5}} = \sqrt[5]{32^4} = \left(\sqrt[5]{32}\right)^4 = 16$ and also that $\left(7^{\frac{2}{3}}\right)^3 = 7^2 = 49$

   Then use the calculator to investigate several other fractional powers in the same way. (Notice on the calculator that SHIFT $x^{\blacksquare}$ gives ($\sqrt[\blacksquare]{\square}$).)

   Check your findings with other students.

Notes for teachers

This module highlights the many different ways in which the CASIO *ClassWiz* can represent numbers, including as decimals, percentages, fractions, in scientific notation and as surds. Powers, roots and reciprocals of numbers are addressed. The module emphasises understanding the representations as well as how the calculator handles them, including prime factorization and making approximations to exact numbers, such as irrational numbers. The text of the module is intended to be read by students and will help them to see how the calculator can be used to deal with various representations of numbers. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together in class and learn from each other.

Answers to exercises

1. 9/20 2. $\frac{149}{99}$ and $1\frac{50}{99}$ 3. 43÷80 = 0.5375 = 53.75% 4. 57% x 16 = 9.12 5. $69 ÷ 15% = $460 6. 2 x 3^2 x 47 x 14593 7. $0.\overline{571428}$ 8. 14/111 9. 117 649 10. 1036 11. It is composite as it has a prime factor of 233 12. 40 353 607 13. Both numbers are 6^6 = 46 656 14. (a) 15625, 390625 and 390625 (b) 2^8 = 256 15. Approximately 3.16227766, which is √10, so its square is 10. 16. 4 checks because 4^5 = 1024 17. 2.8439 18. 72 (which is the square of 6√2.) 19. 2.645751311065 20. [1] [3] [x10^x] [(-)] [6] gives 0.000013 21. 1/49 22. Only (c) π + 1 is irrational

Activities

1. In this activity, students explore the process of efficiently analysing compound interest growth. Encourage them to organise their work to see that the time to double depends only on the interest rate, not on the investment. They might also see the relationship known as the 'Rule of 72', that the time for a compounding investment to double is close to 72÷*R* when interest is *R*% per annum. [Answers: (a), (b) about 14.2 years, (d) about 10.3 years, 9 years, …]

2. Patterns with recurring decimals are easy to generate on the calculator and easy to see for some denominators (such as 3, 7 and 11). They are harder to see for 13ths (since more than one pattern is involved) and very hard to see for 17ths and 19ths, because the recurring part exceeds the width of the calculator display. Encourage students to record results carefully to see the patterns involved.

3. The first part of this activity shows how numbers represented in scientific notation are relatively easy to compare in size. In addition, calculations become easier if students understand the index laws, which help to explain why the patterns work. [Answers: B < A < C < D, C x D = 8.4 x 10^{25}, A + C= 7.7 x 10^{12}, B + C = 4.89 x 10^{12}, D – A = 1.65 x 10^{13} can all be done mentally.]

4. In this activity, the result proposed can be checked by using the calculator, which allows examples to be generated efficiently. [Answer: The result is not true in general; e.g. when $b < 0$]

5. Most students are surprised to find that $5^{-4} > 5^{-5}$, but exploring these sorts of examples on the calculator will help them to appreciate why such relationships hold. It may be wise to remind students to use [SHIFT] [=] to obtain decimal results if they wish to do so, although fractional results are informative with powers of integers. Students may be further surprised to find that $0.8^{-3} < 0.8^{-4}$, but should be encouraged to consider why this happens. (The reason is that 0.8 < 1).

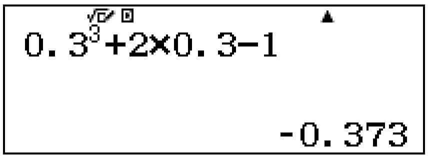
6. The concept of raising numbers to fractional powers is difficult to grasp. This activity uses the calculator to explore some numerical examples to see how powers and roots are involved in the interpretations. Encourage students to examine several examples of their own choosing to see and understand the generalisations involved, especially the key result: $x^{\frac{a}{b}} = \sqrt[b]{x^a}$.

Module 3
Functions

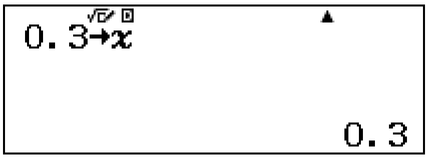
Functions are an important part of mathematics and are widely used to represent relationships of various kinds. The *ClassWiz* is useful for evaluating functions efficiently so that you can examine their properties numerically. These properties include the nature of various kinds of functions (and considering their likely graphs), symmetry properties, asymptotes and relative maximum or minimum values. The calculator can help you to draw and visualise functions, even though it does not have graphics capabilities.

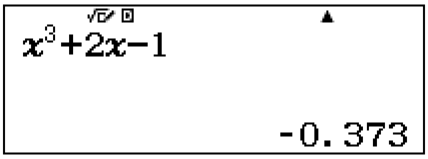
Evaluating expressions and functions

There are several ways of evaluating functions on the calculator in Calculation mode. When an expression or function is to be evaluated at a single point, a calculator can be used directly. Consider, for example, the function $f(x) = x^3 + 2x - 1$. The function can be evaluated for $x = 0.3$ by replacing x by 0.3:



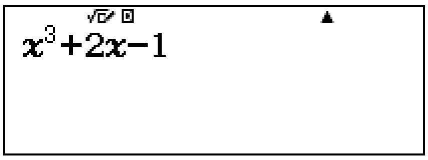
Alternatively, to avoid having to enter the value for x several times, you can store the value in a memory and then evaluate the expression, using [STO] and then [x] as follows:



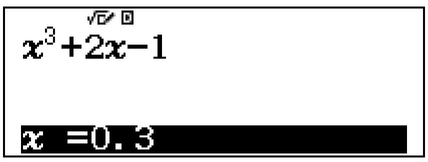


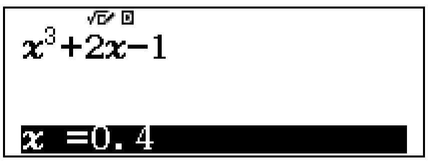
When several different values are involved, however, it is often more efficient to use the CALC capability. For example, suppose you want to evaluate $f(x)$ for $x = 0.3$, 0.4 and 0.5, in order to find which of these values of x is closest to a root of the function, where $f(x) = 0$.

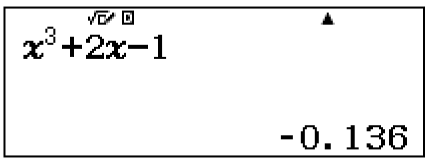
Enter the expression to be evaluated into the calculator, using [x] (but do not tap [=]).



To evaluate the function, tap the [CALC] key, which will result in the calculator displaying the present value for x (in this case, x = 0.3) at the bottom of the screen. Enter the desired value for x, followed by [=]. This will display the new value, as shown in the middle screen below. Finally, tap [=] again to evaluate the expression. The result is shown in the third screen below.

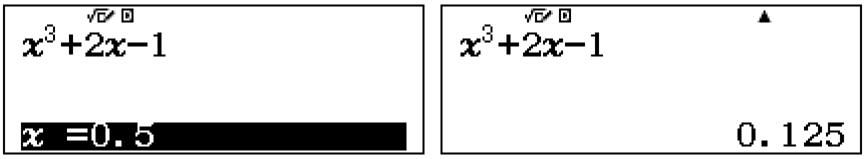






Tap [=] again to continue this process with other values. The CALC process will continue until you tap [AC] or change calculator modes. Many values can be obtained efficiently in this way.

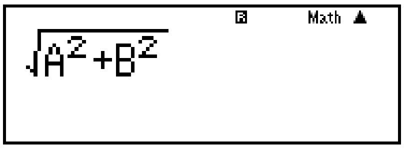
The screens below show $f(0.5) = 0.125$. These three values suggest that, to one decimal place, the value of x closest to a root of the function is $x = 0.5$.



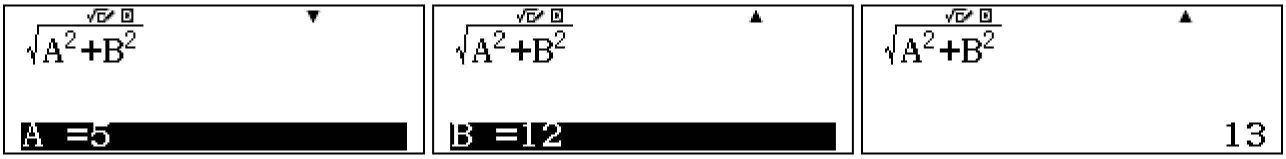
The CALC facility is also useful to evaluate expressions with more than one variable, using the memory keys A to F, x or y. For example, if A and B represent the lengths of the two shorter sides of a right triangle, the hypotenuse is given by

$$\sqrt{A^2 + B^2}$$

Enter this expression into the calculator and tap CALC.



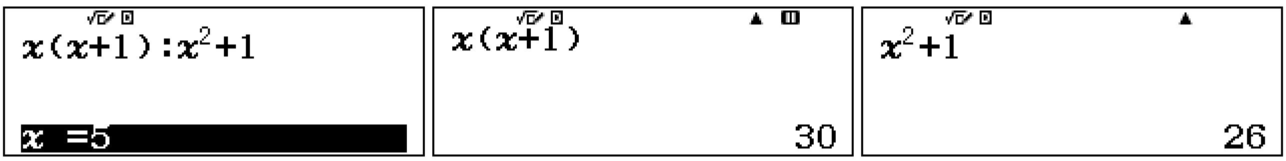
As shown below, the calculator will request values for each of A and B and then evaluate the expression. Tap = after entering each value.



So a right triangle with two shorter sides of length 5 and 12 will have a hypotenuse of length 13. Tap = or CALC again to enter further values for A and B.

Comparing expressions

The CALC facility can also be used to compare several expressions. To do this, enter the expressions, separated by a colon (:) obtained with ALPHA ∫. Then tap CALC and enter values of the variable(s). Each of the expressions will be evaluated in turn by tapping the = key. Consider the example below, comparing values of the two expressions $x(x + 1)$ and $x^2 + 1$.

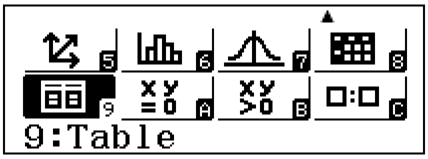


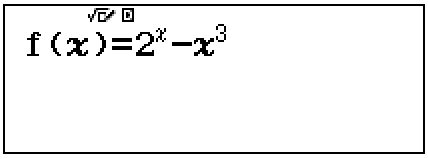
After entering an x value, tap = to get the values of each of the expressions in turn. After the second one, tap = again to enter a fresh x-value. Check for yourself in this way that $x(x + 1)$ has a different value from $x^2 + 1$, except for the single case of $x = 1$.

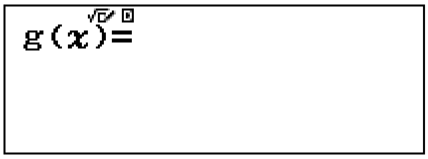
Using tables of values

When a function is to be evaluated at several points, constructing a table of values is often the best option. Use Table mode for doing this. Tap MENU 9 and enter the function, as a function f of x.

For example, to determine where the function $f(x) = 2^x - x^3$ is zero between $x = 0$ and $x = 2$, it would be helpful to evaluate many points and see which is closest to zero. To do so, tabulate the equivalent function $f(x) = 2^x - x^3$, as shown below (as the calculator only uses f for functions):



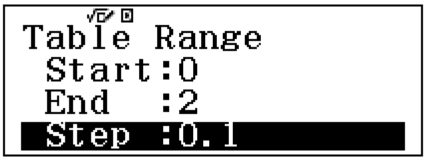


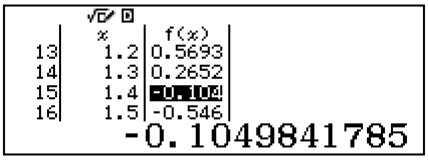


Remember to use the ▶ key to exit from the exponent after entering 2^x.

After the function is entered into the calculator with =, a second function $g(x)$ appears above. As we have only a single function here, this can simply be ignored by tapping =. [It is also possible to use a *Table* command in SET UP to restrict the *ClassWiz* to a single function to be tabulated, instead of two functions, if you wish.]

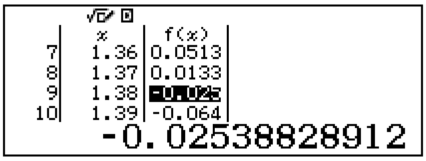
Then you need to specify the *Start*, *End* and *Step* values for the table. These are the first and last values of x to be tabulated, and the increment between them. Importantly, there is a maximum of 30 values permitted in the table, so care is needed. In this case, good choices are shown below. Tap = after each value is entered.





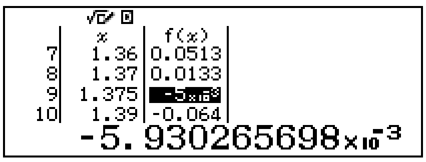
After the *Step* is entered, the table of values will appear. You can scroll the values using ▲ and ▼. It is a good idea to position the cursor in the $f(x)$ column on the right, as the calculator will then display values highlighted with greater precision.

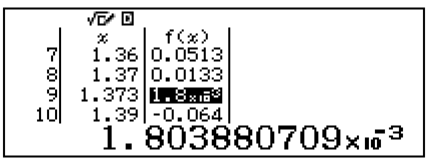
In this case, there appears to be a root between $x = 1.3$ and $x = 1.4$, as the function changes sign between those two points. To increase the precision of the result, construct a new table starting with $x = 1.3$ and ending with $x = 1.4$, but with the smaller step of 0.01. To do so, tap AC (which will *not* clear anything unless you tap it twice) and tap = to enter the new values, as before. Some results are shown below.

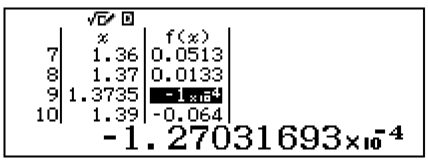


You can repeat this sort of process to increase precision rapidly by one decimal place each time. In this case, it seems that $f(x)$ has a root between $x = 1.37$ and $x = 1.38$.

A good alternative process is available as well. Position the cursor in the x column and replace the highlighted x value by entering a different value. As soon as you tap =, the function is evaluated at the new point. Study the screens below to see a process of 'trial and adjustment', replacing the previous x value of 1.38 each time by 1.375, 1.373 and then 1.3735.



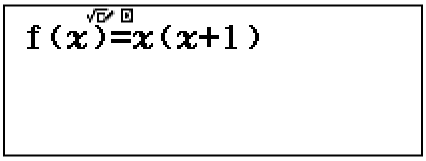


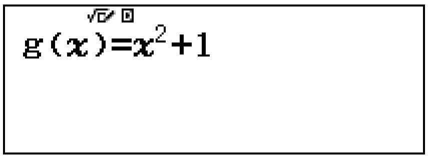


The rest of the table is unaffected by these successive guesses, getting closer to a root of $f(x)$.

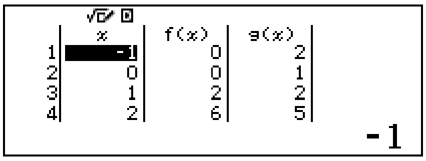
This example demonstrates that a table of values is useful to find important points associated with a function (such as roots, intercepts, turning points, etc.). In the next sections, you will see that tables also allows you to understand other properties of functions.

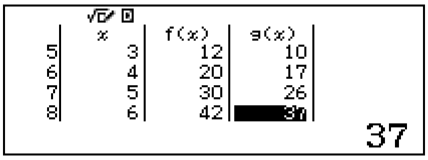
In the previous section, two expressions $x(x + 1)$ and $x^2 + 1$ were compared by evaluating each of them for various values of x. A more efficient process is to generate two tables of values, allowing you to see many values of the expressions quickly. The next two screens show how to set this up in Table mode.

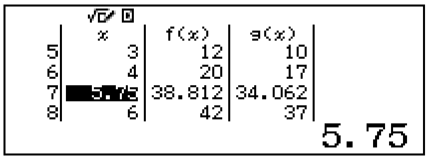




Choose start, end and step values for the tables. The results below show that only when $x = 1$ do the two expressions have the same value. In the third screen, other x values are being tested.







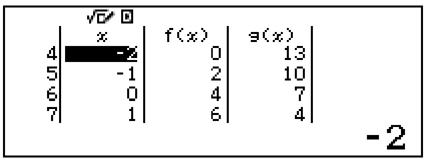
Note that x is the only permissible variable and f and g are the only permissible function names that can be used to enter a function into the *ClassWiz* to generate a table. So functions using other variables need to be entered accordingly. For example, $h(t) = 5t + 7$ needs to be entered as the equivalent function, $f(x) = 2x + 7$. All the numerical values will be the same: only the names of the variable and the function are changed.

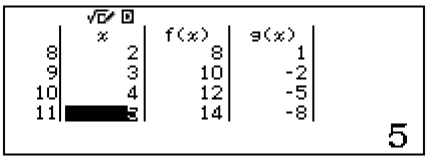
Linear and quadratic functions

A table of values can reveal the nature of families of functions very well. Consider *linear* functions, which include the variable not raised to any power. Here are two examples:

$$f(x) = 2x + 4 \text{ and } g(x) = 7 - 3x.$$

Tables of values of these functions show that they increase or decrease by constant amounts as the variable changes steadily. The two screens below show f on the left and g on the right.

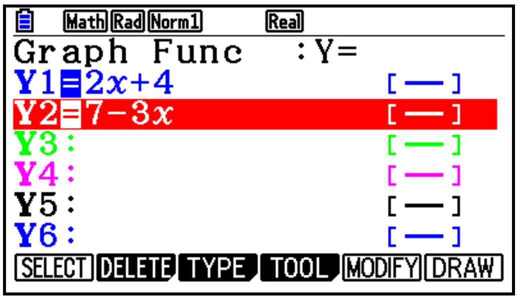


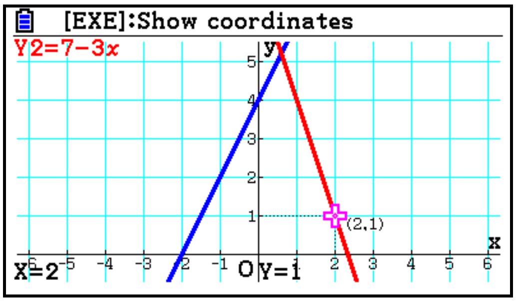


Values for $f(x)$ increase by 2 when x increases by 1, while values for $g(x)$ decrease by 3 when x increases by 1. Scrolling up and down the tables will make this clearer than the screen shots above, which show only four pairs of values each, of course.

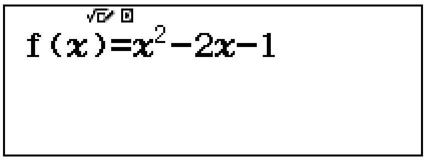
You can use tables like these to obtain values efficiently to draw a graph on paper.

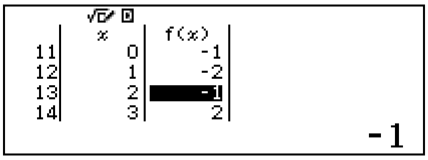
Thinking about linear functions in this way, with the help of the tables, allows you to *imagine* the graphs of the functions: $f(x)$ is increasing with a slope of 2 and $g(x)$ is decreasing more sharply with a slope of -3. As the following graphs (drawn on a CASIO fx-CG20 graphics calculator) show, each function can be represented graphically as a line, which is why they are called *linear* functions.





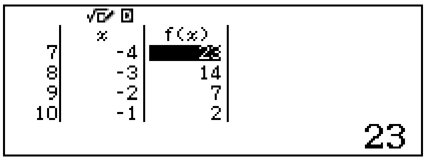
Quadratic functions include a variable raised to the second power, and have a different character from linear functions: the value of the function does not change steadily when the value of the variable changes steadily. For example, consider the quadratic function $f(x) = x^2 - 2x - 1$, which has been tabulated below. As x increases from 1 to 2, the value of the function increases by 1: from -2 to -1. But as x changes from 2 to 3, the value of the function increases by 3: from -1 to 2. Unlike the linear function, the increase is not steady.

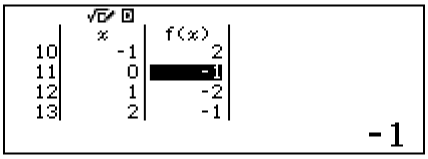


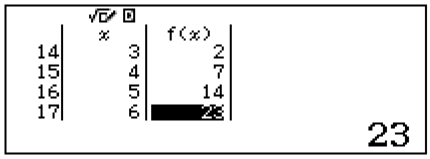


Scrolling the table also reveals that the values of the function increase to the right of $x = 1$ and to the left of $x = 1$ in the same way: in fact the values of the function are identical either side of $x = 1$, which has the lowest value with $f(x) = -2$. Study the table segments below to verify this for yourself.

Again, tables like these can be used to help you sketch on paper a graph of the function efficiently.



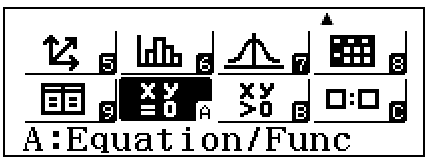


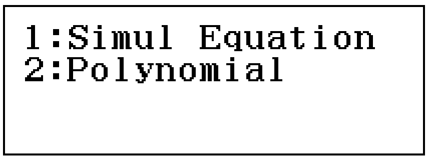


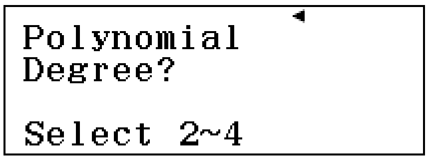
Starting with the minimum value of $f(1) = -2$, the values of the function increase by 1, 3, 5, 7, etc when the value of x changes by 1 in each direction. (Notice that these *increases* are linear: they increase steadily by 2 each time, characteristic of quadratic functions whose x^2 coefficient is 1.)

Although it is helpful to use a graph to understand the nature of this function, a table of values can help you to visualise the shape and characteristics of the graph of a function.

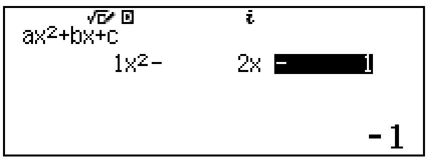
While a table of values is helpful, and allows you to sketch a graph, the *ClassWiz* can provide even more information to understand the graph of a quadratic function. Change to *Equation/Function* mode using MENU A, and choose a polynomial function of degree 2, as shown below.



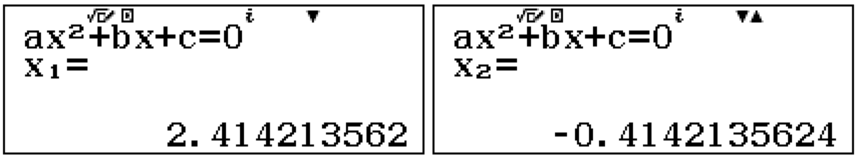




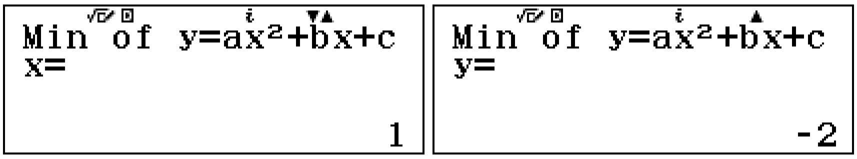
Enter the three coefficients of the quadratic function, 1, -2 and -1, tapping = after each one.



Then tap ≡ to obtain each of the two roots of the function, x_1 and x_2, as shown below.

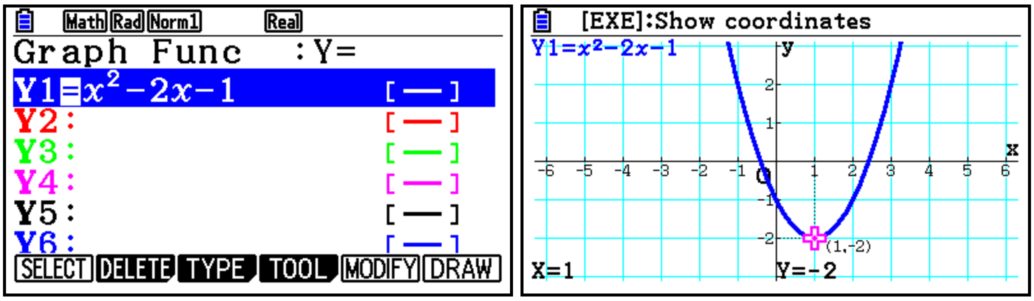


Finally, tapping ≡ twice will provide the coordinates of the turning point (1,-2) of the parabola:



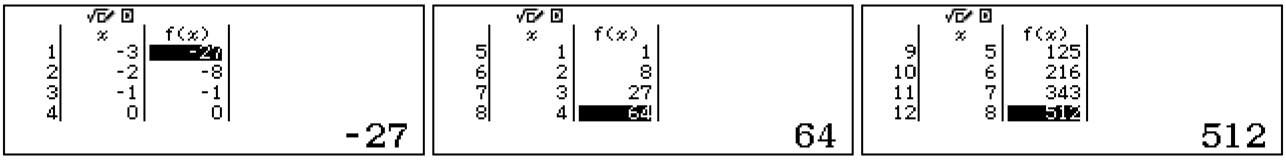
Together with the values in the table, these values of the roots and the turning point will allow you to produce a good sketch of the graph of the quadratic function on paper.

As you can see from the CASIO fx-CG20 screens below, the properties described above are all visible in a graph, which shows that the parabola has a line of symmetry through the turning point at $x = 1$:

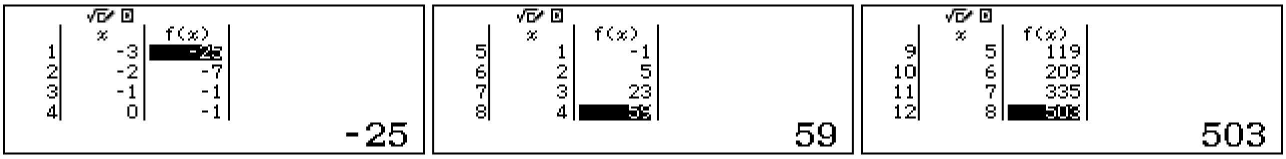


Cubic functions

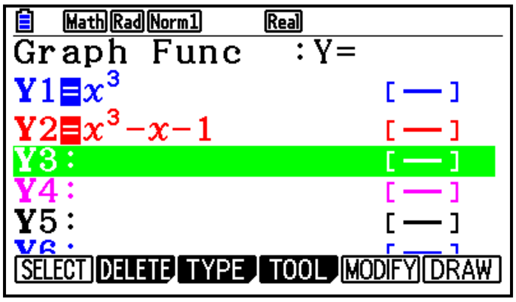
Cubic functions have a term with the variable raised to the third power. As for linear and quadratic functions, a table of values can be helpful to understand the nature of a particular cubic function. For example, the table segments below suggest that the cubic function $f(x) = x^3$ increases, but not in a linear way, and has negative values for $x < 0$ and positive values for $x > 0$.

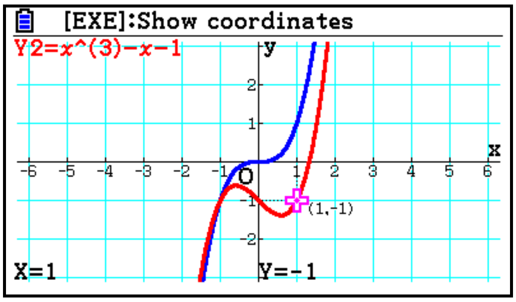


While this (basic) cubic function is symmetrical about the origin ($x = 0$), other cubic functions are not. Again, a table of values will offer some insights into the shape of the function. The tables below show a different cubic function $f(x) = x^3 - x - 1$, that is *not* symmetrical about the origin.



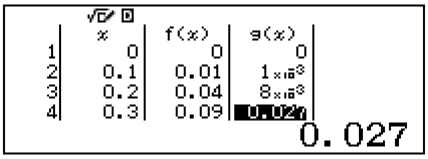
These two sets of tables are helpful to visualise the shapes of the two cubic functions, represented in the graphs below, drawn with a CASIO fx-CG20 graphics calculator. These graphs make it clear that there are various shapes for cubic functions (unlike linear functions, which have only a linear shape and quadratic functions, which have only a parabolic shape).

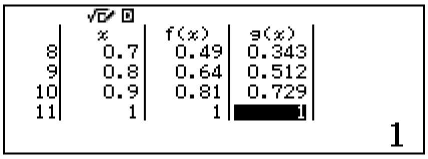




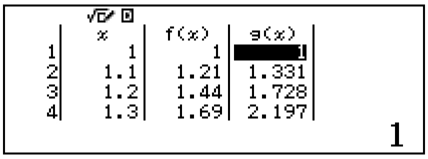
Tables can also help you to compare quadratic and cubic functions. For example, consider the two functions $f(x) = x^2$ and $g(x) = x^3$. If you look closely at tables of values, over the same domain, you will see that the cubic function increases more slowly than the quadratic function for $0 < x < 0.5$ (for example) and more sharply for $1 < x < 2$.

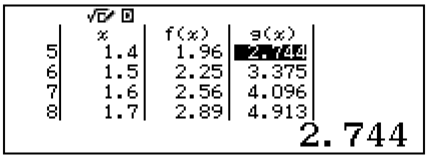
To see these differences consider these table values for $0 < x < 1$:

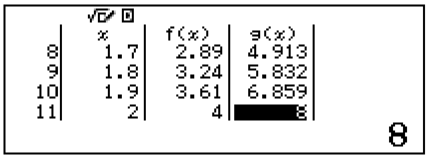




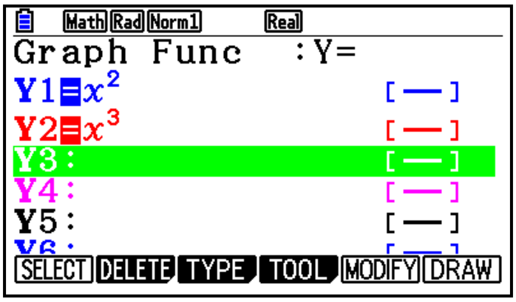
Notice the different behaviours in the tables below for $1 < x < 2$. The cubic function is now increasing faster than the quadratic function in this case.

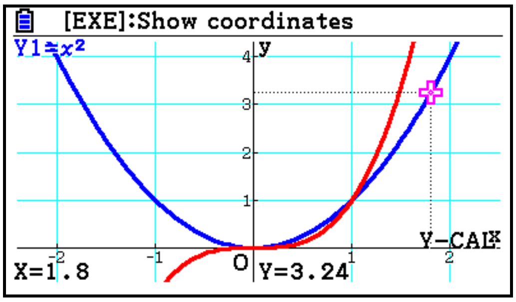






Both types of functions are curves, rather than lines, but the curves are different from each other. The graphs below reflect these characteristics as well.





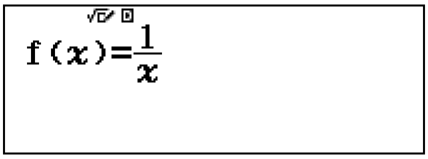
Again, studying the tables of values for these functions allows you to understand closely the ways in which these functions change as the values of the variables change, as well as to sketch graphs.

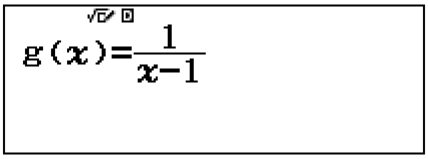
Reciprocal functions

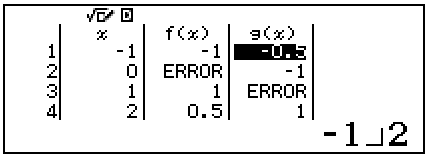
Reciprocal functions, which involve division by a variable, can be studied in similar ways. We will consider two examples of reciprocal functions:

$$f(x) = \frac{1}{x} \text{ and } g(x) = \frac{1}{x-1}$$

One particular property is immediately clear if you construct a table of values for which the denominator of the function has a zero value. In this case, the table for $f(x)$ includes $x = 0$ and the table for $g(x)$ includes $x = 1$ (so that $x - 1 = 0$):



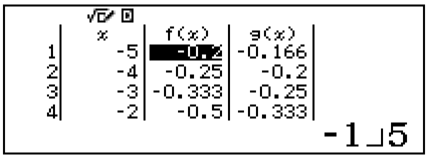


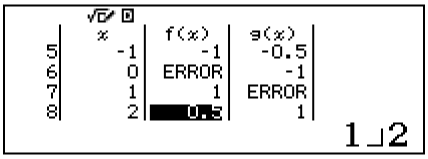


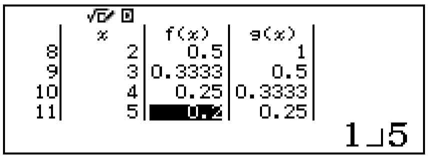
Because division by zero is undefined, there is no value of the function at $f(0)$ and $g(1)$, so the calculator gives an error in each case.

Related to this kind of discontinuity is another difference concerning the nature of reciprocal functions: that their values do not change smoothly and continuously like other functions considered so far in this module, but 'jump' sharply, so that graphs of these function have two distinct parts.

Considering again the reciprocal function shown above, inspection of some tabulated values shows this phenomenon. Examine carefully the screens below:

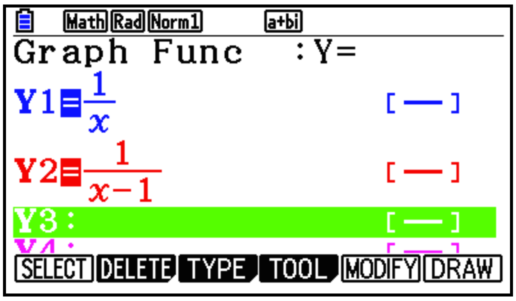


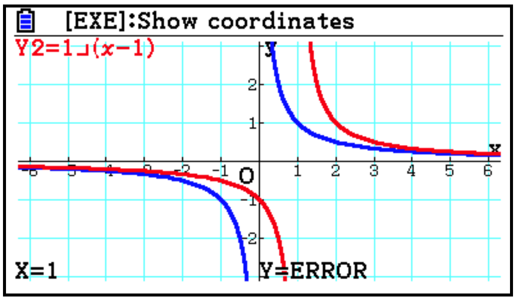




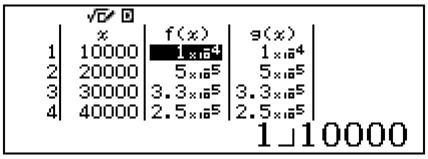
Studying the tabulated values like this reveals that $f(x)$ has positive values for $x > 0$ and negative values for $x < 0$, while $g(x)$ has positive values for $x > 1$ and negative values for $x < 1$. It is also clear that the values for $f(x)$ either side of zero are opposites of each other, suggesting that the function is symmetrical around $x = 0$. The values for $g(x)$ shows that it is symmetrical around $x = 1$. Tables like these will allow you to sketch graphs of the functions on paper.

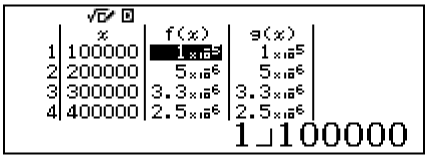
Graphs of these two function, such as those below drawn on a CASIO fx-CG20, show these kinds of properties well:

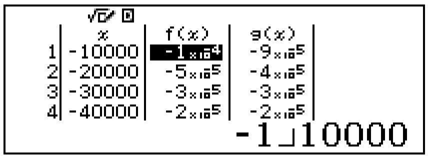




Another property of the reciprocal function is that the values seem to get closer and closer to $y = 0$ for both large positive and large negative values of x. However, the values never reach zero: those with $x > 0$ are always positive, while those with $x < 0$ are always negative. You can see this by tabulating some very large values. The examples below illustrate this:



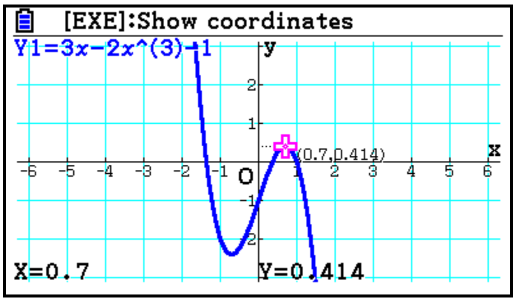




For these particular functions, there is an *asymptote* of $y = 0$. The function values get closer and closer to 0, but do not ever reach it. Asymptotes are important in advanced mathematics, and you are most likely to see them with reciprocal functions.

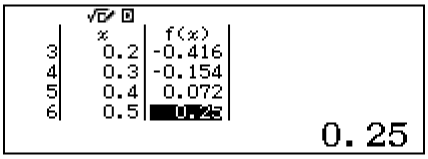
Maximum and minimum values

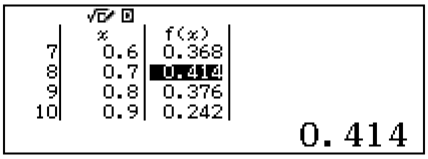
You may have noticed that some functions seem to have maximum values or a minimum values at some points. Quadratic functions have either a maximum value or a minimum value, while many cubic functions have *both* of these, represented by 'loops' in the graph of the function. For example, the graph of $f(x) = 3x - 2x^3 - 1$ below (drawn on a CASIO fx-CG20) seems to reach a maximum value between $x = 0$ and $x = 1$ and a minimum value between $x = -1$ and $x = 0$.



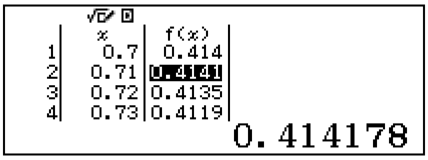
Notice that these values are only 'maximum' or 'minimum' values in a restricted or relative sense: for example, the value of the function for x < -2 is much larger than any values of the function between $x = 0$ and $x = 1$. There is a *local maximum* in this case between $x = 0$ and $x = 1$.

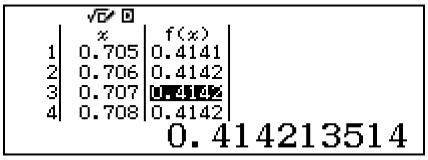
A table of values is very convenient to study maximum values or minimum values on a small interval. In the case of $f(x) = 3x - 2x^3 - 1$, tabulating the values between $x = 0$ and $x = 1$ is helpful:

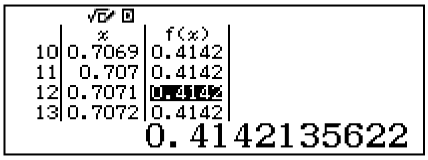




It seems from the table that there is a maximum value of the function between $x = 0.6$ and $x = 0.8$. To examine the maximum value more closely, choose smaller and smaller intervals, with correspondingly smaller step sizes. Scroll the table to look for a maximum value. The screens below show this process being used repeatedly, with step sizes of 0.01, 0.001 and 0.0001 respectively:

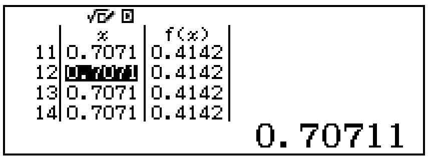


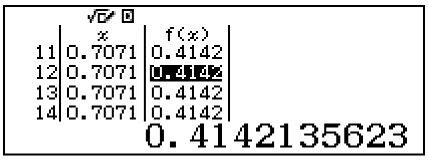




Notice that all the tabulated values in the final screen above seem to be the same (0.4142), because the table can display only four decimal places. However, better precision is available by scrolling with the cursor in the $f(x)$ column.

The next two screens show a further zoom in, using a step of 0.00001. Notice that *both* columns now seem to be unchanging, because of the table size limitation. However, scrolling in either column reveals more precise values.





For this function, there seems to be a (local) maximum value of $f(x) \approx 0.41421$ near $x \approx 0.7071$. If you continue to tabulate the function on increasingly smaller intervals in this way, you can get a more accurate result.

An *exact* result, however, is available only through the use of calculus; the *ClassWiz* can be used only to find good numerical approximations. In this case, the exact value (determined by other means) is exactly $\sqrt{2} - 1$, when $x = \sqrt{2}/2$, so that the numerical procedure gives a very good approximation after just a few steps.

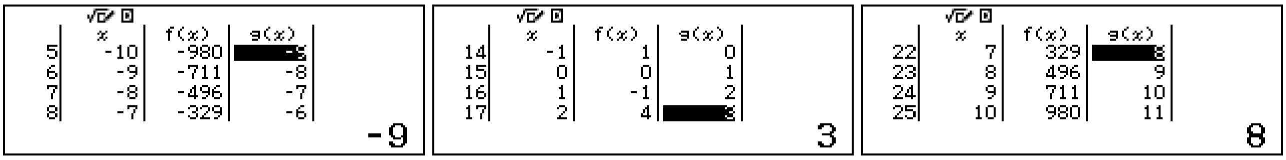
Similar kinds of processes of repeated zooming in a table allow you to find the relative minimum values of a function. Check for yourself that the function $f(x)$ studied here also has a relative minimum value of about -2.4142 close to $x \approx -0.4142$.

Intersection of two graphs

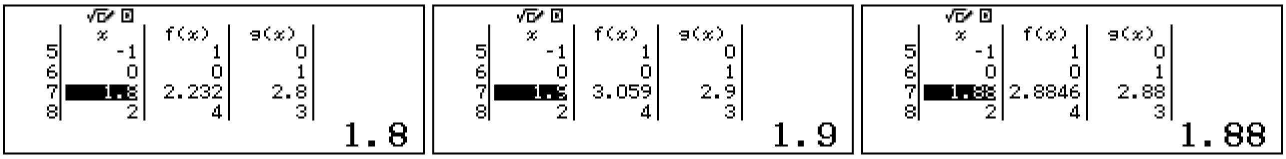
As already noted above, the calculator allows you to visualise mathematical ideas such as graphs of functions, even though it does not have a capability to draw the graphs. Another good example of this involves the intersection of the graphs of functions. Consider the following two functions:

$$f(x) = x^3 - 2x$$
$$g(x) = 1 + x$$

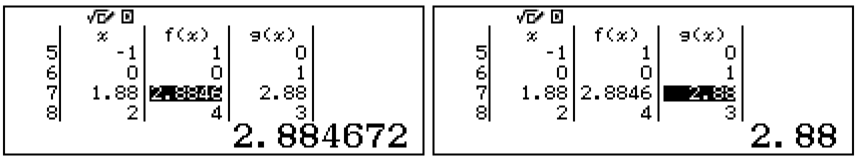
The graphs of the functions intersect when they have a point in common: the same x-value and y-value. Tabulating some values suggests that the function values are close to each other near the origin, for $-1 < x < 2$.



There is a point of intersection between $x = 1$ and $x = 2$. To approximate this point, move the cursor to the x column (with $x = 1$) and enter a good guess for the value of x for which $f(x)$ and $g(x)$ will be the same, and tap ⊟. The first screen shows that a guess of $x = 1.8$ is a little small, with $f(x) < g(x)$.



The second screen suggests that $x = 1.9$ is a little large, with $f(x) > g(x)$. Our third screen, following some further guesses shows that $x = 1.88$ is quite close, although a little small. Notice below that $f(1.88)$ and $g(1.88)$ are very close, suggesting that the graphs intersect near $x = 1.88$.



Notice that these are not precisely the same (because the solution is only an approximation):

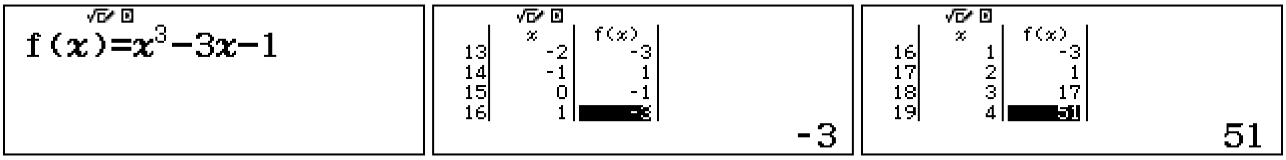
$$f(1.88) = 1.88^3 - 2 \text{ x } 1.88 = 2.884672$$
$$g(1.88) = 1 + 1.88 = 2.88$$

So, a good approximation to the points of intersection of the graphs is (1.88,2.88).

Another way to explore points of intersection is to consider when a single difference function is zero, rather than compare two functions. So, tabulate instead the difference function $d(x)$:

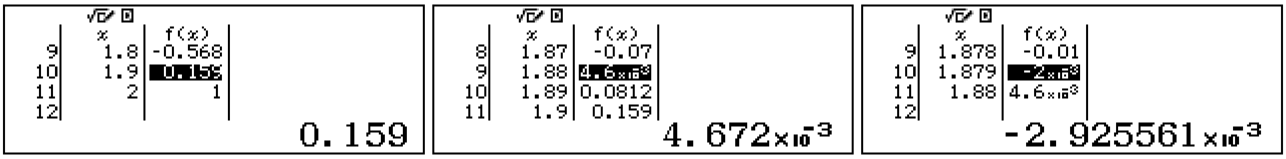
$$d(x) = f(x) - g(x) = x^3 - 3x - 1 \ .$$

The graphs intersect at the points for which the difference function is zero. With limited information about where the graphs might intersect, a good choice to start with is a table of values from $x = -14$ to $x = 15$ with a step of 1. Scrolling this table to find points for which $f(x)$ is close to zero reveals three possible intervals, as shown below:



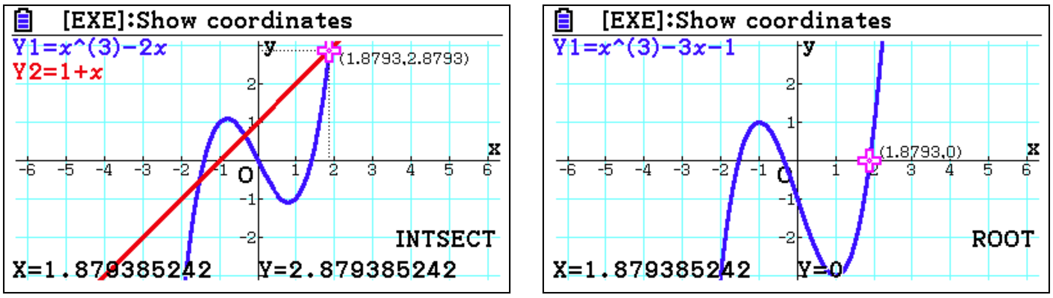
When the cubic function changes sign (either from negative to positive or vice versa), it will pass through zero. So there appear to be zero points in $-2 < x < -1$, $-1 < x < 0$ and $1 < x < 2$. We will examine just the last of these three intervals here.

Zooming in on the table values repeatedly allows you to get an increasingly accurate approximation to the solution. Study carefully the extracts from the sequence of three tables below, which use ten values in each of the successive intervals $1 < x < 2$, $1.8 < x < 1.9$ and $1.87 < x < 1.88$.



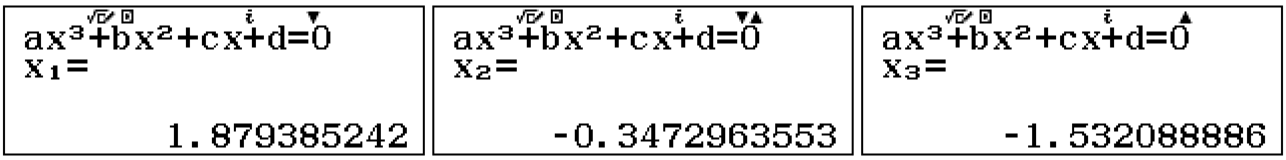
Notice that the step size for the tables is smaller each time, 0.1, 0.01 and 0.001 respectively, so that in effect, the calculator allows you to get an increased decimal place of accuracy with each successive table. You could continue this process much longer, but we will choose to stop here, suggesting that $x \approx 1.88$ is a good approximation, correct to two places of decimals, consistent with the first approach used.

The graphs below, drawn on a CASIO fx-CG20, show how the intersection of graphs of two functions can be imagined as finding the roots of a single function.



Notice that one of the points of intersection points is close to the values obtained above and that there is a root of the difference function close to $x = 1.88$.

In Module 4, you will see that the *ClassWiz* can be used in this case to automatically find all three solutions to the cubic equation $x^3 - 3x - 1 = 0$, using Equation mode, as shown below.



However, the procedures described here using tables of values can be used for other pairs of functions that cannot be handled in Equation mode, and also allow you to see better how the function values are related.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. (a) Store 0.3 in the x memory in order to evaluate $f(x) = x^3 + 4x + 1$ for $x = 0.3$.

   (b) Evaluate $f(0.4)$ by first editing the store command in (a) to make $x = 0.4$.

   (c) Evaluate $f(0.5)$

2. Use the CALC facility to evaluate $f(x) = x^3 + 4x + 1$ for $x = 0.6$, 0.7 and 0.8.

3. Evaluate $g(x) = 1.2x^2 + 3.1x + 2.7$ for $x = 1.1$, 1.2 and 1.3.

4. The longest diagonal of a rectangular room with dimensions A, B and C is given by the expression $\sqrt{A^2 + B^2 + C^2}$. Use the CALC facility to find this diagonal length for a room with dimensions 4.2 m x 7.3 m x 2.1 m and another room with dimensions 3.2 m x 3.5 m x 2.1 m.

5. Make a table of values for the linear functions $f(x) = 17 – 2x$ and $g(x) = 4x – 3$, starting with $x = 1$ and ending with $x = 25$, increasing in steps of 1. Then use your table to:

   (a) find $f(14)$ and $g(19)$.

   (b) find the values of x for which $g(x) = 93$ and for which $f(x) = -7$.

   (c) describe how the values of the functions are changing as x increases.

6. Make a table of values for the quadratic function $y = 3 – x – x^2$ for values of x between $x = -1.5$ and $x = 1$ in steps of 0.1. Use your table to

   (a) find $f(0.3)$

   (b) find the maximum value of the function

   (c) find the value of x for which the function has its maximum value

   (d) determine which is larger, $f(0.8)$ or $f(0.9)$

   (e) find the value(s) of the function for which $f(x) = 2.25$

   (f) describe the intervals on which the function is increasing or decreasing

7. Use a suitable table of values for the function $f(x) = x^3 – 3x^2 + 2x$ to find

   (a) the values of x for which $f(x) = 0$

   (b) the values of x for which $f(x) < 0$

   (c) the maximum value of $f(x)$ on the interval $0 \le x \le 1$.

8. Consider the function $f(x) = \dfrac{4}{x-5}$. Use a suitable table of values to find

   (a) any points of discontinuity

   (b) the values of x for which $f(x) < 0$.

9. Find the point of intersection of the graphs $y = x^3$ and $y = x + 2$, correct to two decimal places.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. Make some tables of values for the function $f(x) = 2x + 7$ and $g(x) = 5 - 4x$.

   (a) Scroll tables with a step of 1 to see how the values of $f(x)$ and $g(x)$ change as x increases.

   (b) Scroll tables with a step of 0.5 to see how the values of $f(x)$ and $g(x)$ change as x increases.

   (c) Scroll some tables with different steps and compare your observations with those in parts (a) and (b)

   (d) Change the functions to $f(x) = 2x + 5$ and $g(x) = 3 - 4x$ and compare your observations with those in parts (a), (b) and (c) above.

2. Study some other linear functions such as $f(x) = 5x + 7$ and $g(x) = 6 - 2x$. Compare the behaviour of the functions with your observations from Activity 1.

3. Tabulate the quadratic function $f(x) = (x + 1)(x - 3)$ from $x = -14$ to $x = 15$, with a step of 1. Notice that some values are the same (e.g., $f(-4) = f(6)$). Study these carefully to see which pairs are the same. Notice that the values of $f(x)$ do not change the same amount each time the value of x is changed (unlike the case for linear functions). Use the values in the table to sketch a graph of the function on paper. Which aspects of the graph could you predict from the table?

4. A small rocket is launched in the air from the ground. Its height after t seconds is given by the quadratic function, $h(t) = 30t - 4.9t^2$.

   (a) Use a table of values for this function to find the maximum height reached by the rocket and the time at which this occurs; it will be helpful to use the table values to sketch a graph.

   (b) After how many seconds does the rocket return to the ground?

5. Consider the cubic function, given by $y = x^3 - 2x - 1$.

   (a) Use a suitable table of values to see that there is a root of the function at $x = -1$ and another root between $x = 1$ and $x = 2$.

   (b) Construct tables of values between $x = 1$ and $x = 2$ to find the root as accurately as possible.

   (c) There is another root of the function near $x = -1$. Use a similar process of successive tables to find this root as accurately as you can. A graph may help you to do this efficiently.

   (d) Use the CALC facility with the expression $x^3 - 2x - 1$ to check how close the approximate roots you have found are to the actual roots.

6. Reciprocal functions have a point of discontinuity, for which their value is not defined. You can find the associated x-value from a table. If you study the function carefully, you should be able to predict these points in advance. Try this process for the following functions: predict where the function will be discontinuous and then make a table of values that verifies your prediction:

   (a) $f(x) = \dfrac{3}{x}$ (b) $f(x) = \dfrac{4}{x-5}$ (c) $f(x) = \dfrac{1}{2x+3}$ (d) $f(x) = \dfrac{2}{3x-4}$

   Make up some more examples of this kind and explore their properties.

Notes for teachers

This module highlights the ways in which the *ClassWiz* can support students to think about elementary functions (linear, quadratic, cubic and reciprocal) through evaluating a function at a point or in tables of values. The module makes considerable use of Table mode as an important tool. Although some graphs of functions are shown (drawn on a CASIO fx-CG20 graphics calculator), it is important to realise that *ClassWiz* does not include a graphics capability, so the emphasis is on visualising the functions and determining their properties by careful choice of tables. The text of the module is intended to be read by students and will help them to see how the calculator can be used to examine various kinds of functions. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. (a) 2.227 (b) 2.664 (c) 3.125 2. 3.616, 4.143, 4.712 3. 7.562, 8.148, 8.758 4. 8.68 m and 5.19 m 5. (a) -21, 73 (b) 24, 12 (c) As x increases by 1, $f(x)$ decreases by 2, $g(x)$ increases by 4. 6. (a) 2.61 (b) 3.25 (c) -0.5 (d) $f(0.8) = 1.56$ (e) $x = -1.5$ and $x = 0.5$ (f) Function is increasing for $x < -0.5$ and decreasing for $x > -0.5$ 7. (a) Tabulate $f(x)$ in steps of 1 to find x = 0, 1 or 2 (b) Tabulate $f(x)$ in steps of 0.5 to find $x < 0$ and $1 < x < 2$ (c) Tabulate $f(x)$ with successively smaller steps to find $f(0.423) \approx 0.385$ 8. Use steps of 1 to find (a) $x = 5$ (b) $x < 5$ 9. (1.52,3.52)

Activities

1. The main purpose of this activity is for students to see that linear functions increase or decrease in a steady way, represented by the slope of the function. It is assumed that students might not yet have been formally introduced to the concept of slope, so that this kind of activity will help them to appreciate the idea with a close look at an example. Sketching graphs on graph paper from the tables is also a useful and appropriate activity.

2. This activity has a similar purpose to Activity 1, but deliberately includes less direction to students. The two functions have a positive and a negative slope, and you should encourage students to try some other examples for themselves.

3. This and the next activity are concerned with quadratic functions, and a key idea is for students to understand the nature of quadratic relationships. In this activity, the symmetry of the relationship is emphasised. Encourage students to sketch some graphs on paper. [Answers: The intersections with the x-axis at $x = -1$ and $x = 3$ can be predicted from the form of the function, although students may well be surprised with the graph. These points help to predict the line of symmetry ($x = 1$).]

4. This activity provides a context as well as mathematical explorations, and gives substantive meaning to the idea of a maximum and a root. [Answers: The rocket reaches a height of about 45.9 m after 3.06 seconds and returns to the ground after 6.12 seconds.]

5. This activity emphasises the idea of 'zooming' in a table to refine values and get better approximations. Students will be able to get good approximations to the roots near 1.618 and -0.618 but these will not yield a zero value with the check in (d). The exact roots are at $x = (1 \pm \sqrt{5})/2$ and can lead to fruitful discussions about exact and approximate values. You may choose to give these values to students to check and to distinguish the ideas of approximate and exact values.

6. In this activity, attention is focussed on the tables that give an 'Error', as division by zero is not defined. Students may not be familiar with the formal definition of discontinuity, so that a graph of a reciprocal function (such as that in this module's text) may be helpful. Construction of the four tables increases in difficulty. (In part (c), the step must include halves. In part (d), it must include thirds; enter the step as 1 ⊟ 3). [Answers: (a) $x = 0$ (b) $x = 5$ (c) $x = -3/2$ (d) $x = 4/3$]

Module 4
Equations and inequalities

Equations and inequalities are important parts of algebra. The *ClassWiz* offers a number of different ways of solving them. On this calculator, approximate numerical solutions to equations can be found; exact solutions require either algebraic analysis or a computer algebra system.

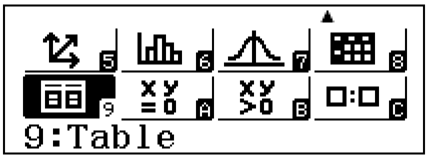
In this module, we will use Table mode, Equation/Function mode, Inequality mode, Ratio mode and the CALC and SOLVE functions on the calculator keyboard. Make sure your calculator is set into *MathIO* mode for both input and output and *Norm 2*. Use SET UP to do this, if necessary.

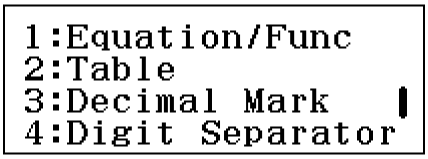
Equations, inequalities and a table

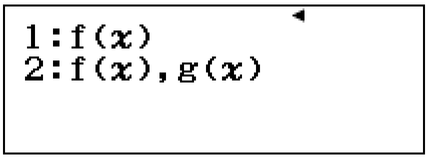
The basic idea of solving an equation in one variable is to find which values of the variable, if any, make the equation true. Once solutions to an equation are found, associated inequalities can also be solved. There are a few ways to solve equations using a table to evaluate a suitable function.

Consider for example the equation $x^3 = 3x^2 - 1$. The solutions to this equation are also the solutions to the related equation $x^3 - 3x^2 + 1 = 0$. So we will use the calculator to tabulate this function to see for which values of x, if any, it has a value of zero.

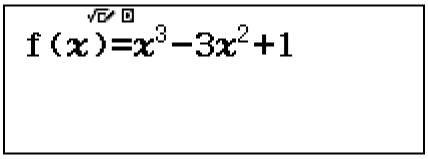
Access Table mode with MENU 9. The *ClassWiz* allows you to tabulate one or two functions. When you need only one function, you can choose to tabulate only one function using SET UP, as shown below, or you can tabulate two functions, but simply clear any definitions of $g(x)$.

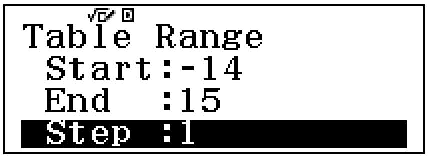




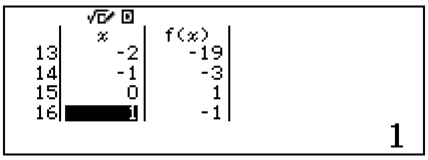


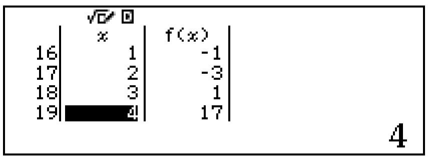
Enter the function $f(x) = x^3 - 3x^2 + 1$, as shown below. (Note that there is an x^3 command, SHIFT x^2.) Tabulate this function over a suitable range. Start by tapping =. In this case, without any information about solutions, it is wise to use a wide range to start. As only 30 values are permitted, we will choose values for x from -14 to 15 inclusive, as shown below. Tap = after each value in the Table Range screen.





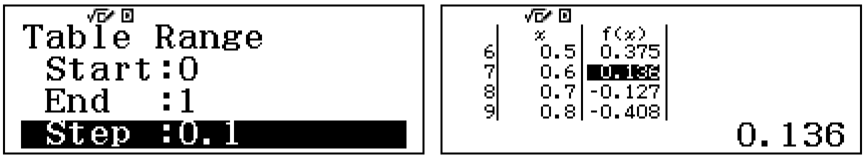
Use the cursors ▲ and ▼ to scroll the second column, looking for values that are zero, or close to zero. Look also to see if there are places where the values change between positive and negative, as these suggest that there will be a value in between with a value of zero.



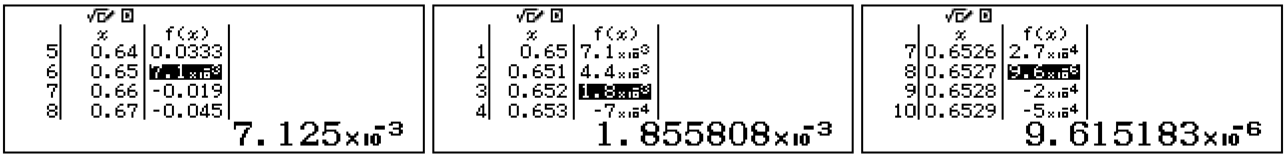


In this case, there seem to be no values in the table for which $f(x) = 0$. However, there are sign changes in the intervals $-1 < x < 0$, $0 < x < 1$ and $2 < x < 3$, indicating that the function has roots in each of these three intervals. We will explore only one of these for now: the interval $0 < x < 1$.

Tap AC to return to the function definition and zoom in a little closer on this interval by setting *Start* to 0, *End* to 1 and *Step* to 0.1.



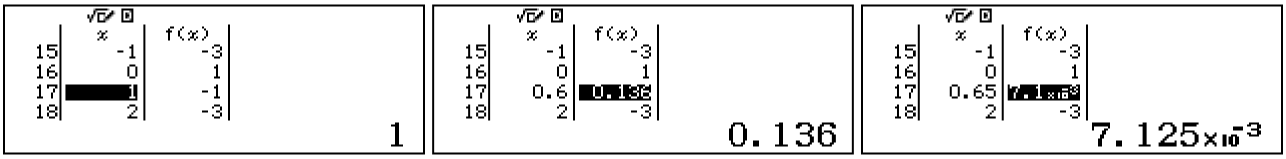
Study this table carefully. When $x = 0.6$ the function has a positive value. When $x = 0.7$, the functional value is negative, so it must be zero somewhere between these two values of x. So repeat the process of making a table, this time using the interval [0.6, 0.7] for which $0.6 \leq x \leq 0.7$ and a smaller step of 0.01. The first screen below shows the result.



Each time you repeat this process, the interval becomes smaller, as the next two screens above show, using intervals of [0.65,0.66] and [0.652,0.653] respectively. Each time, the step is reduced by a factor of ten, so the result is closer to the actual root. The third screen shows that the function has a zero between $x = 0.6527$ and $x = 0.6528$.

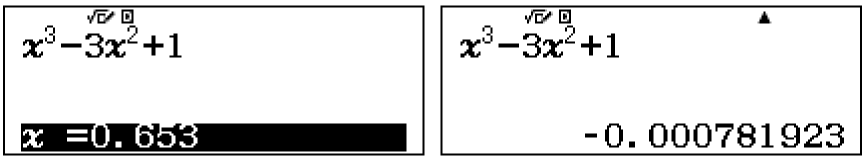
You will need to decide for yourself on a suitable level of accuracy. These tables show that there is a root of the function at $x \approx 0.653$. So $x \approx 0.653$ is close to a solution of the equation. You could continue this process to get an even closer approximation to solutions, but, as they are irrational numbers, you will not be able to obtain an *exact* solution.

There is another way you can use the table to find an approximate solution to the equation. Start with the screen on the left below, obtained originally. Use the keyboard to enter approximations to a possible solution in the x-column, followed by = to evaluate $f(x)$ each time. Two examples are shown below in the next two screens, with $x = 0.6$ and then $x = 0.65$.



Continue this process of changing the value of x until the value of $f(x)$ is close enough to zero.

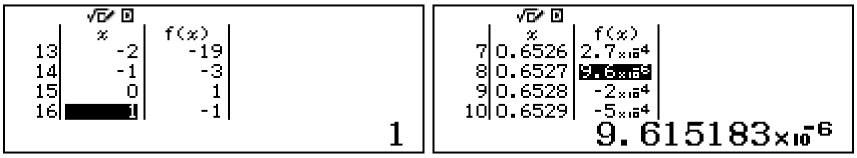
You can check how close your solution is to an exact solution by seeing how close $x^3 - 3x^2 + 1$ is to zero when $x = 0.653$. A good way to do this is to return to COMP mode and enter the expression in the screen as shown below. Then tap the CALC key and enter the value of x as 0.653. Tap = to see the value.



When $x = 0.653$, the expression has a value of -0.000781923, which is very close to 0, so this value of x is close to a solution to the equation. If you wish, you can tap = and test some other values in the same way; you can see that $x = 0.6527$ is even closer to a solution to the equation. (But do not tap AC unless you wish to delete the expression.)

There are two other solutions to this equation, as noted above. You should use these same procedures to find them for yourself. As a check, you should find $x \approx -0.532$ and $x \approx 2.879$.

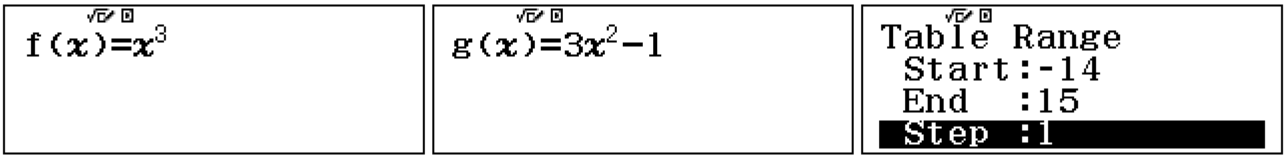
Once equations have been solved, inequalities can be solved. For example, the inequality $x^3 < 3x^2 - 1$ can be rewritten as $x^3 - 3x^2 + 1 < 0$. The tables used to solve the equation make clear some values for x for which the function is negative, such as those below:



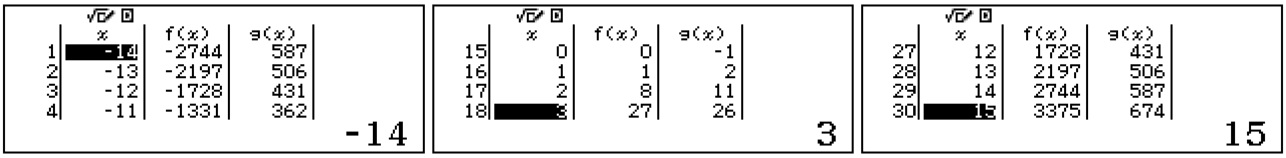
Once the solutions to the equation are found, the solutions to the inequalities can be written down.

Using two tables

Similar ideas can be used to solve equations and inequalities with two tables, with a separate function associated with each side of an equation. Consider again the same equation, $x^3 = 3x^2 - 1$.



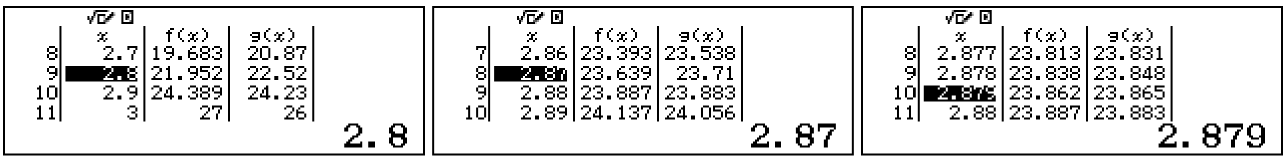
Study the tables carefully. Here are some extracts:



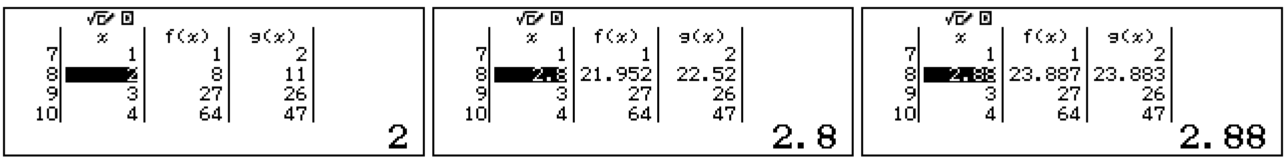
The first screen shows many values of x for which $f(x) < g(x)$, or $x^3 < 3x^2 - 1$. Similarly, the third screen shows many values of x for which $f(x) > g(x)$, or $x^3 > 3x^2 - 1$.

The situation is more complicated in the middle screen, however, since $x^3 > 3x^2 - 1$ when $x = 0$ and $x = 3$, but $x^3 < 3x^2 - 1$ when $x = 1$ and $x = 2$.

Similar procedures to those in the previous section can be used to find good approximations to the solutions to the equation. For example, the successive screens below allow you to find a good approximation to the solution of the equation between $x = 2$ and $x = 3$, by looking for values of x that have $f(x) = g(x)$; that is, the values in the two columns are equal. Continue this process yourself for at least one more step.



The alternative approach is to adjust the x-value successively to get closer and closer to values for which $f(x) = g(x)$. Here are some possible screens:

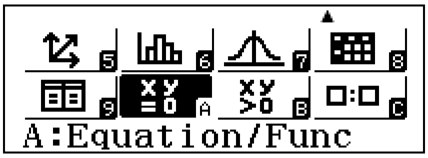


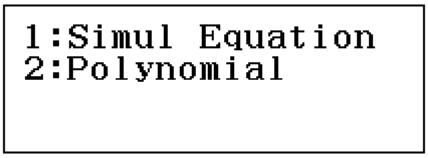
Continue this process yourself to get a better approximation to the x-value for which $f(x) = g(x)$.

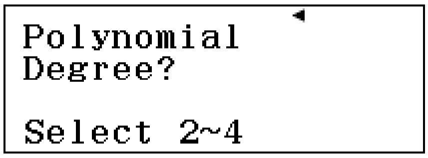
Once good approximations to solutions have been found, you can use these to write down solutions to associated inequalities. In this case, for example, it seems that $x \approx 2.88$ is one solution to the equation $x^3 = 3x^2 - 1$. Then, using the table to help your thinking, it seems that part of the solution to $x^3 > 3x^2 - 1$ is $x > 2.88$.

Automatic equation solving

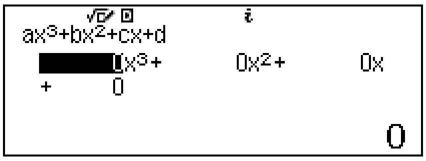
The procedures above are important because they emphasises the meaning of a solution to an equation or an inequality. However, there are faster and more efficient ways of solving cubic equations and inequalities on this calculator. Use MENU A to choose *Equation/Function* mode. Then tap 2 to select polynomial equations followed by 3 to select cubic equations, as below:

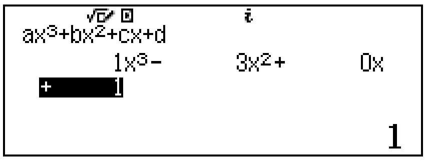




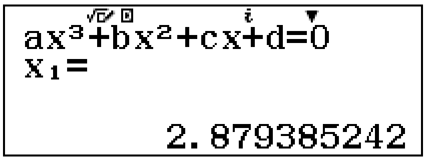


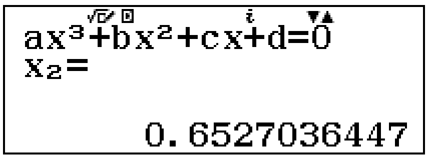
You need to enter the coefficients a, b, c and d of the equation $x^3 - 3x^2 + 1 = 0$ in the order shown below. In this case, the coefficients are 1, -3, 0 and 1 respectively. Tap = after entering each one.

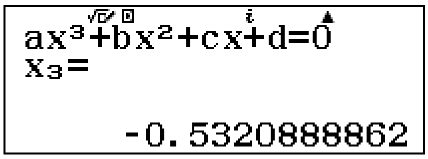




To solve the equation now, simply tap = again, to give the three roots in turn, shown below.



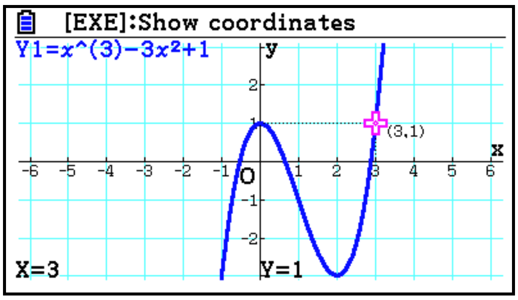




Notice that the first solution is consistent with the one found using two tables in the previous section, while the second solution is consistent with that found from a single table. While these are better approximations than those found using tables above, they are still *approximate* solutions.

If you tap AC, you will return to the coefficients, and if you tap it a second time, you will clear the coefficients.

In this case, the cubic equation has three real solutions. You could use the values obtained earlier to sketch a graph of the function used in the single table, $f(x) = x^3 - 3x^2 + 1$. This would help to see why the equation has three solutions. The graph below, drawn with a CASIO fx CG-20 shows that the solutions to the equation are the roots of the function – the values of x where the graph crosses the x-axis.



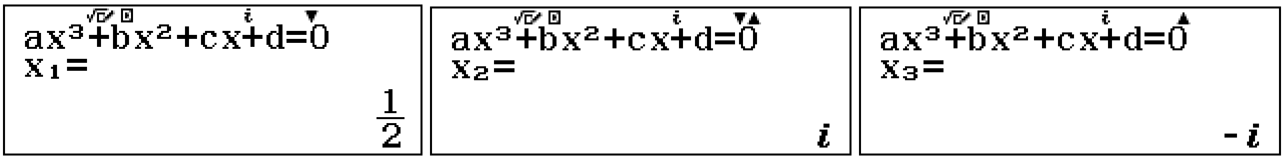
Cubic equations will always have three solutions, but sometimes some of the solutions are complex numbers, which are discussed in Module 9. As an illustration, consider the equation

$$2x^3 + 2x = x^2 + 1$$

To enter the coefficients for this equation into the calculator, you must firstly rearrange it to the required form, in descending order of powers of x, with $a = 2$, $b = -1$, $c = 2$ and $d = -1$:

$$2x^3 - x^2 + 2x - 1 = 0$$

Enter this equation into the calculator, and tap $\boxed{=}$ to check that the three roots are as shown below.

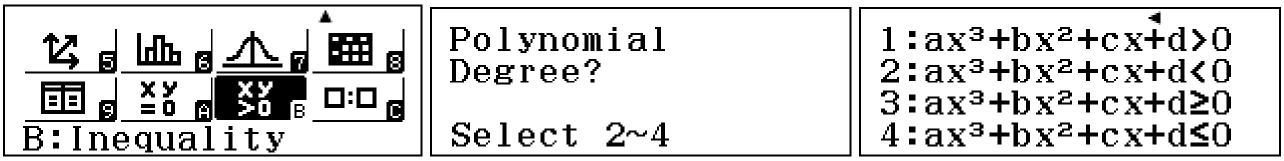


In this case, two of the roots are complex numbers, $x = \boldsymbol{i}$ and $x = -\boldsymbol{i}$.

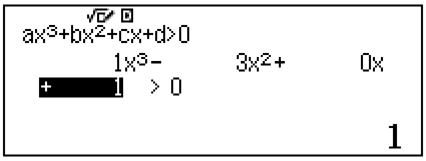
Automatic inequality solving

As noted above, you could use the solutions to the equation to solve associated inequalities. However, the calculator allows you to solve inequalities automatically, too. Consider the inequality studied before: $x^3 > 3x^2 - 1$.

Use $\boxed{\text{MENU}}$B to enter Inequality mode. Select $\boxed{3}$ for the highest power of the inequality and then 1 to choose the form that matches the inequality of interest (>, in this case).

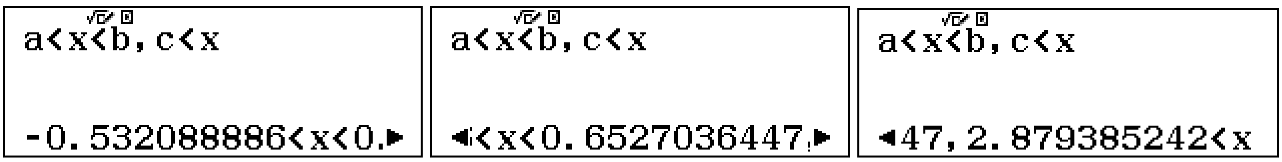


It is necessary to think about the inequality to fit the calculator form. In this case, write $x^3 > 3x^2 - 1$ as $x^3 - 3x^2 + 1 > 0$. Enter the coefficients $a = 1$, $b = -3$, $c = 0$ and $d = 1$ as shown below:



Tap $\boxed{=}$ to obtain the solutions. The results $a < x < b$, $c <$ x indicate that there are two intervals for which $x^3 - 3x^2 + 1 > 0$. [Note that a, b and c are *not* the coefficients of the original inequality, but are used to show the form of the solution.]

In this case, the solution does not fit on the screen, shown by the arrow on the right below. To see the solution in full, you will need to use the cursor keys ▶ and ◀, shown below.



Correct to three decimal places, the solutions are $-0.532 < x < 0.653$, $x > 2.879$. These are the values of x for which the graph above is above the x-axis and thus $x^3 - 3x^2 + 1 > 0$, or $x^3 > 3x^2 - 1$.

Solving quadratic equations and inequalities

There is an automatic quadratic equation solver also available in Equation mode; it works in the same way as the cubic solver, requiring you to enter coefficients in descending order of powers of x. Quadratic equations have two solutions, not three as for cubic equations, and they may also be in the form of complex numbers.

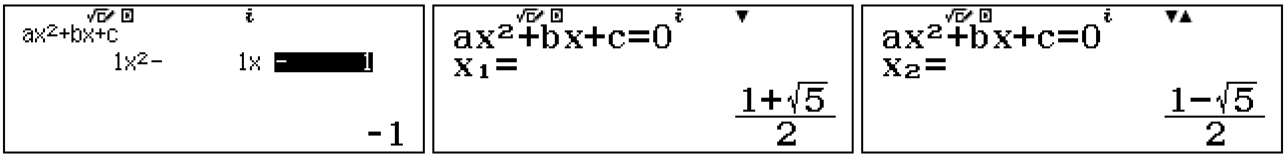
To illustrate the process, consider the problem of finding a number that can be squared by adding one to itself. If the number is represented by n, the equation representing this property is $n^2 = n + 1$. To solve this equation in the calculator's quadratic equation solver, you need to rearrange it into the standard form shown on the calculator:

$$ax^2 + bx + c = 0$$

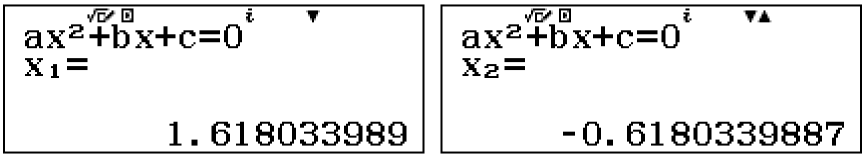
In this case, the rearranged equation is $n^2 - n - 1 = 0$, which is equivalent to

$$x^2 - x - 1 = 0.$$

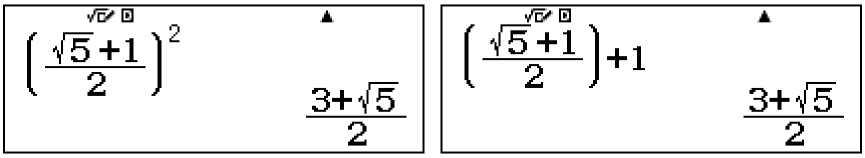
In Equation mode, select type [2] and enter the three coefficients: $a = 1$, $b = -1$ and $c = -1$.



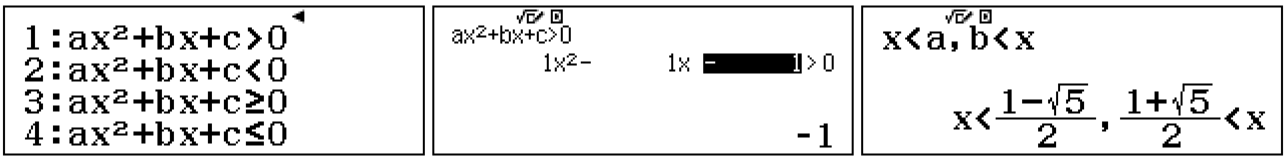
The two solutions are irrational, but are shown above in exact form (since the *ClassWiz* is set to MathIO.) Decimal approximations can be obtained using the [S⇔D] key:



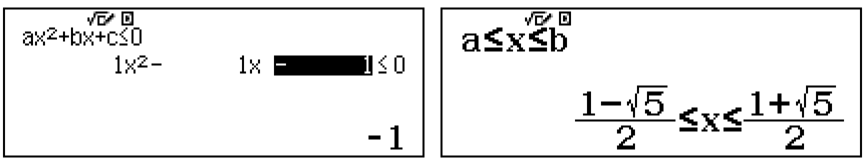
One of the solutions in this case is the Golden Ratio, much admired by the Greeks. As the screens below show this solution satisfies the required property $x^2 = x + 1$.



Similarly, quadratic inequalities can be solved in Inequality mode, obtained with [MENU] B. To solve the inequality $x^2 > x + 1$, first rewrite it as $x^2 - x - 1 > 0$. The solution is obtained directly:

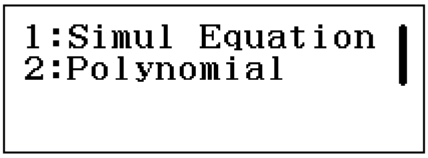


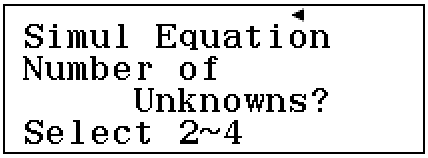
This solution comprises two separate (open) intervals. The related inequality, $x^2 - x - 1 \leq 0$, is satisfied on a single interval, for values of x between the same two values:



Systems of linear equations

Some equations involve more than one variable. In Equation mode, systems of *simultaneous linear* equations in two, three or four variables can be solved. After selecting simultaneous equations with [1], you need to select the number of simultaneous equations:





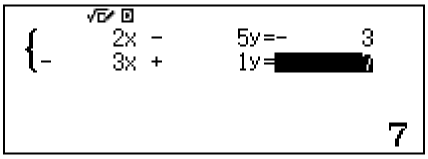
The process is similar in each case, so we will consider a two-variable pair of equations:

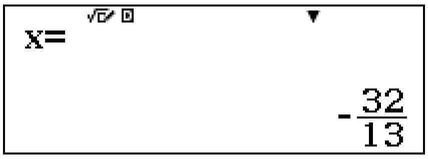
$$2a + 3 = 5b$$
$$b - 3a = 7$$

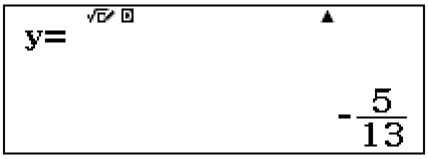
In order to enter the system into the calculator, you need to think of the variables as x and y instead of a and b and also to rewrite the equations with the constant terms on the right side of the = sign. Making each of these two changes produces the equivalent system of equations:

$$2x - 5y = -3$$
$$-3x + y = 7$$

Select the two-variable case by tapping [2] and enter the six coefficients as shown below in the appropriate spaces. Make sure that you use the [(−)] key for the negative signs. Tap [=] after each coefficient is entered. Use the cursor keys to backspace if necessary to correct any typing errors. Solutions to the equation can now be obtained by tapping [=] again, once for each variable:





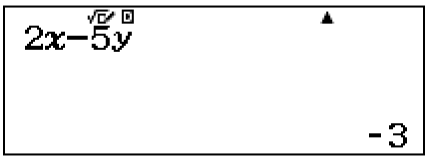


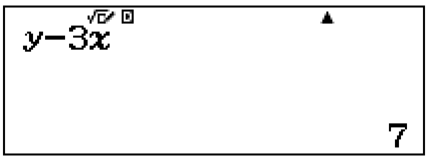
If a decimal result is preferred, tap the [S⇔D] key after each solution is given.

You can check that the solutions of $x = -32/13$ and $y = -5/13$ fit the original pair of equations by substitution. Usually it is best to do this by hand. As the values are a little awkward to manipulate in this case, we describe an efficient way to do this in Calculation mode, after first storing the two solutions into the x and y memories in Equation mode.

When the solution for x is found, store it immediately in the x memory with [STO] [x]. Similarly, when the solution for y is obtained, store it immediately in the y memory with [STO] [S⇔D].

Then switch to Calculation mode with [MENU] [1]. Evaluate the left sides of the two equations to see that these two values for x and y do indeed satisfy *each* of the two equations, as shown below:



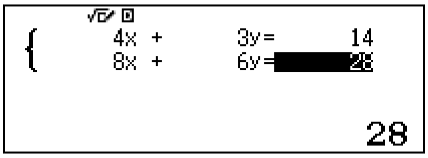


For these values of x and y, $2x - 5y = -3$ and $y - 3x = 7$, as required by the simultaneous equations.

You need to be careful with systems of linear equations, as they do not always have a solution. Consider for example the following system.

$$4x + 3y = 14$$
$$8x + 6y = 28$$

When you attempt to solve this system on the calculator, the result suggests that there is an infinite solution, as shown below.

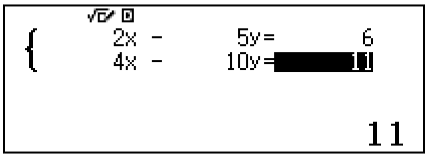


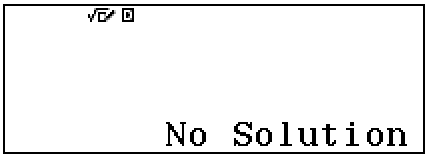
In this case the two equations are actually equivalent versions of the same equation. Notice that the second equation is ‘double’ the first equation, so any pair of values that satisfy the first equation will also satisfy the second equation. For example, $x = 2$ and $y = 2$ satisfy the equations, but so also do $x = 5$ and $y = -2$, as well as $x = 8$ and $y = -6$, and an infinite number of other solutions. If $4x + 3y$ equals 14, as the first equation demands, then $2(4x + 3y) = 8x + 6y$ will be equal to 28, as the second equation states. Equations like these are described as *dependent*, and do not have a unique solution.

There is another way in which a system of linear equations might have no solution. Here is an example:

$$2x - 5y = 6$$
$$4x - 10y = 11$$

When you solve this system on the calculator, no solution is found::





In this case, the two equations are not equivalent, as they were for the dependent case above. But if you study them carefully, you will see the problem. If $2x - 5y = 6$, then $2(2x - 5y) = 4x - 10y$ must be equal to $2 \times 6 = 12$. Yet the second equation in this system is $4x - 10y = 11$, which is inconsistent with the first equation. Systems of equations like this are called *inconsistent*, and have no solution.

Using the Solver

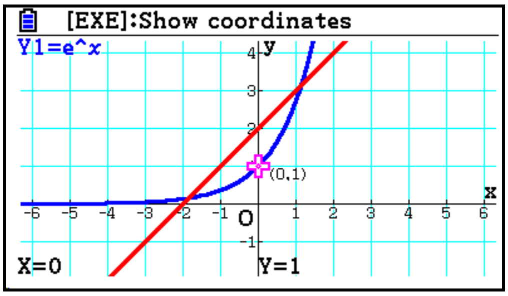
There are many other kinds of equations than simultaneous linear equations, quadratic and cubic equations. On the calculator, a *Solver* is available to find a numerical solution to any equation that has only one variable, which is usually x.

Consider, for example, the equation

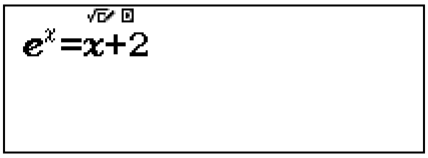
$$e^x = x + 2$$

This equation does not fit any of the four categories of equations that are addressed in Equation mode. Yet it can be solved (approximately) by the calculator.

The meaning of the equation is helped by imagining (or sketching quickly) a corresponding graph. In this case, the following graph (produced with a CASIO fx-CG20 graphics calculator) suggests that there are two solutions to the equation, corresponding to the two points where the graphs of $y = e^x$ and $y = x + 2$ intersect.

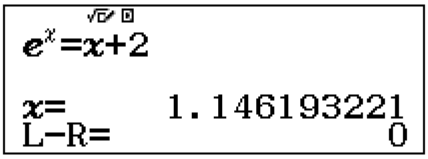


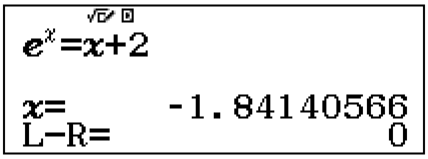
In Calculate mode of the calculator, enter the equation as shown below, using [ALPHA] [CALC] for the equals sign (that is, *not* using the normal [=] sign on the calculator.). Don't forget to use [▶] to exit from the exponent of *e*. Don't tap the [=] key when you have finished.



To begin the process of solving this equation, enter the *solve* command using [SHIFT] [CALC]. The calculator will ask for an initial guess for the value of *x*, as shown below. (The value showing on the screen (0 in the case below) is not significant; it is merely the most recent value for *x* used in the calculator.)

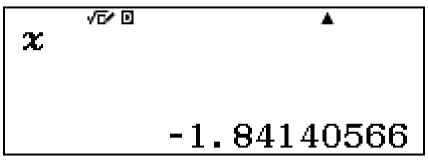
Enter a choice that you expect to be near a solution and tap [=]. For this equation, the graph above suggests that there are solutions near $x = 1$ and $x = -2$. When 1 is entered, followed by the [=] sign, the calculator finds its first solution, as shown on the left below, after a few seconds. Tap [=] again to enter another starting value for the search for a solution. In this case, -2 is a good choice. Tap [=] to get the second screen shown below.

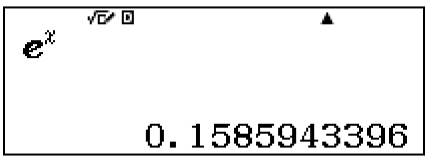


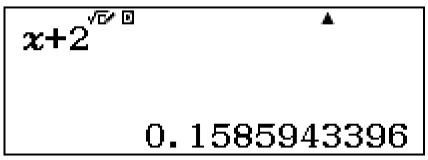


Each of these screens shows a solution to the equation, expressed to the accuracy permitted by the display: $x \approx 1.146193221$ and $x \approx -1.84140566$. In each case, the expression L – R evaluates the difference between the Left and Right sides of the equation, and in each case indicates that the two sides are equal, since L – R = 0, to the accuracy of the display.

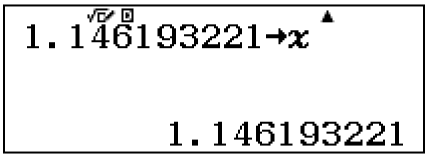
You can check either or both of these results by evaluating the expressions in the calculator. Tap [AC] to start. The second solution is easy to handle, as the calculator has already stored the most recent value of *x* in the *x* memory, which you can recall with the [x] key, as shown below:

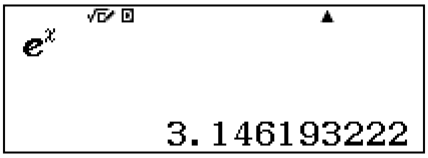


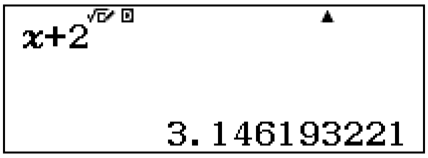




The last two screens do show that for x = -1.84140566, then $e^x = x + 2$. It is a little more tedious to check the first solution, as the value for x has to be entered by hand first. To avoid entering it twice, you can store it in the x memory first.



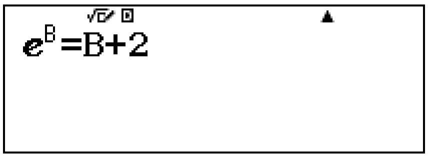


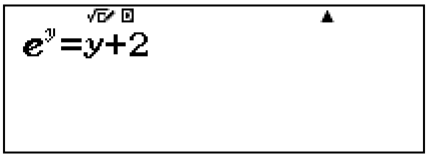


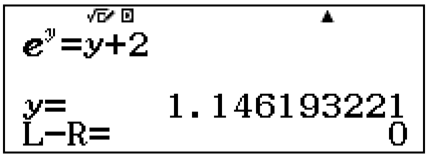
In this case, the two approximate values differ only in the final decimal digit (because the value used for x is only an approximation to the solution found by the *ClassWiz*. It would be unlikely that a solution to more than a few decimal places was appropriate, however, so that approximate solutions such as $x \approx 1.1462$ and $x \approx -1.8414$ are quite adequate for most practical purposes.

You may wonder what the calculator is doing to solve the equation. It is using similar processes to those we used with tables earlier in this module, of making an estimate, checking the result and then using the data to improve the estimate … an *iterative* process. The calculator processes are faster than those done by hand, of course.

Equations can be solved for different variables from x. Check for yourself that solving these two equations gives the same values as the version using x as the variable. One of the solutions for y is shown. The meaning of the equation is not changed by changing the variable.

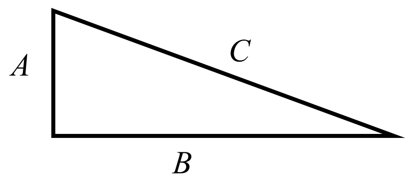






The solver function in the calculator can solve an equation for only one variable. So if an equation involves more than one variable, the calculator will require you to give a numerical value for any other variables before finding a solution.

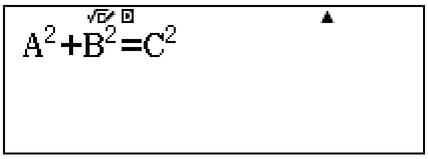
Consider the example of finding the length of one side (A) of a right triangle, if the length of the hypotenuse (C) and another side (B) are known.



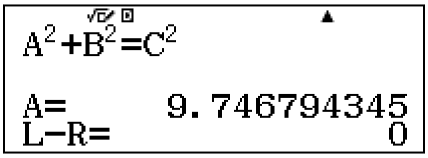
The relationship between these three variables, A, B and C is the Pythagorean relationship:

$$A^2 + B^2 = C^2.$$

To find the value of A when B = 7 and C =12, enter the equation into the calculator, again using [ALPHA] [CALC] for the equals sign.



Now solve the equation using [SHIFT] [CALC]. When a variable is showing, you can assign a value with the keyboard. If you tap [=] without assigning a value, the calculator will find a solution for that variable. In this case, the calculator needs values for B and C, in order to solve for A. You can use the ▲ and ▼ keys to move between variables. Input values for each of the other two variables, followed by the = key. Give A a suitable starting value (if you think the value showing is not already suitable). After B and C have values, move to A and tap = to solve the equation. The solution of $A \approx 9.75$ is given below.



Make sure that the calculator starts with a reasonable guess for the variable of interest. In this case, for example, giving a negative value as an initial guess for A will lead to a negative solution, which does not make sense for the context of finding lengths of sides of a triangle.

If you now wish to solve further triangles, for differing values of A, B and C, start by tapping the = key and then repeating the process.

This use of the solver is very helpful if you wish to solve a number of equations of the same kind, such as those that use a similar formula.

Ratio and proportion

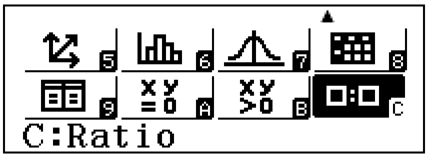
A common form of equation involves quantities that are in *direct* proportion to each other. Consider, for example, a car travelling at a constant speed. If it travels 28 km in 20 minutes, how long would it take to travel 45 km? The ratio of the distances travelled is the same as the ratios of the times taken. So, if x represents the required time, the proportion can be stated using ratios as:

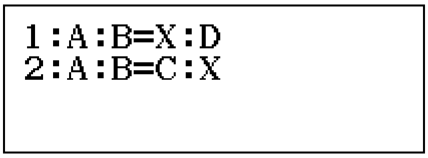
$$28:45 = 20:x$$

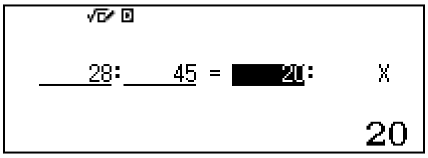
The ratio of the distances is equal to the ratio of the times. You can also write this as an equation using fractions:

$$\frac{28}{45} = \frac{20}{x}$$

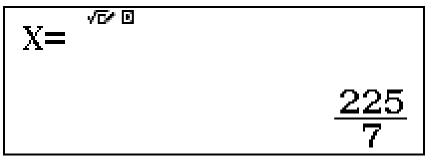
To solve equations like this, use *Ratio* mode, (MENU C) as shown below. Choose the appropriate form, in this case 2, then enter the three values involved, tapping = after each, as shown below.

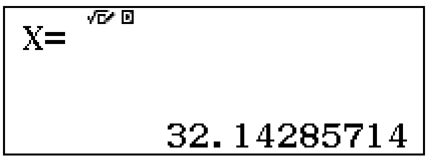






Tap = to obtain the answer, which can be expressed as a decimal, using the S⇔D key if you wish. Note that the solution of a little more than 32 minutes seems reasonable for this problem, as you would expect the car to take longer to travel 45 km than it did to travel 28 km.





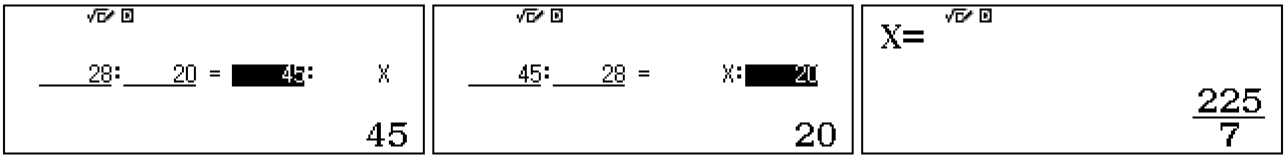
Tap = again to enter different values to solve another proportion of this same type. To change to ratios of the other type, tap OPTN and select the type.

In this case, you might have chosen to represent the direct proportion in different ways. For example, the following are the alternatives, shown as ratios or fractions

$$28:20 = 45:x \qquad\qquad 45:28 = x:20$$

$$\frac{28}{20} = \frac{45}{x} \qquad\qquad \frac{45}{28} = \frac{x}{20}$$

These are all equivalent representations. The first one shows that the ratio of distance to time (i.e., the speed) is constant, while the second shows that the ratios of the distances and the times are the same, but is expressed differently from before, in Type 1 for the *ClassWiz*. As you can see by solving them in Ratio mode, the solution for x is the same in each of the three cases.



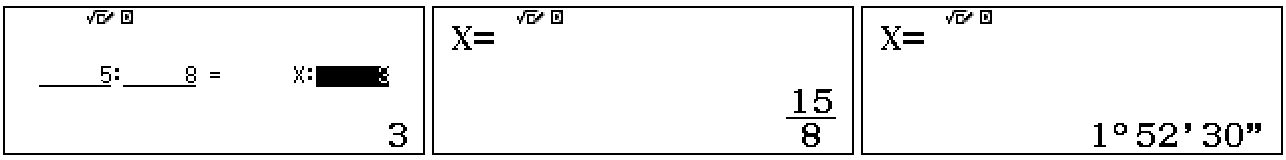
Some relationships are in *inverse* proportion, not direct proportion, so care is needed in writing the ratios. A typical example is the following:

> *Five women working together take three hours to build a wall. How long would eight women take to build the same kind of wall?*

In this case, increasing the number of women *decreases* the time required, so the ratio equations to find the time x are

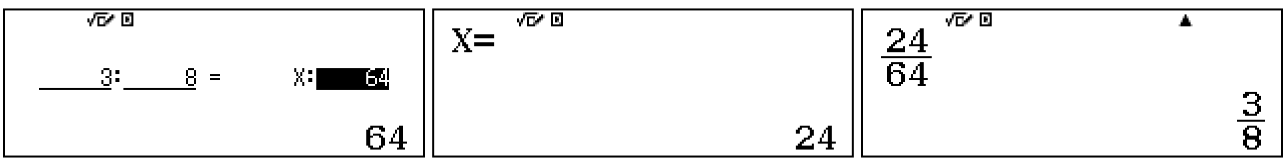
$$5:8 = x:3 \qquad \text{or} \qquad \frac{5}{8} = \frac{x}{3}$$

The solution is available in Ratio mode, as before:



The result here has been converted to hours, minutes and seconds (using the °’” key), although such precision is not really warranted in this context. Importantly, notice that the answer makes sense in the context: you would expect eight women to take less time than five woment to do the same job, so a solution of less than two hours seems reasonable here.

Ratio mode can be used for other purposes, to understand relationships. For example, suppose you wanted to represent a fraction with a different denominator. For example, what fraction with 64 in the denominator is equivalent to three eighths? This can be represented as a ratio problem of $3:8 = x:64$ and solved using the *ClassWiz*.



The result of $x = 24$ can easily be checked in Calculate mode by entering the fraction 24/64, which the calculator represents as 3/8, consistent with the result above. Of course, you are expected to be able to make conversions of these kinds mentally, as it is usually more efficient to do so.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. Use the *solve* function to solve $6x + 19 = 46$.

2. (a) Use Equation mode to solve for x: $x^2 - 7 = x$. Give solutions to 2 decimal places.
   (b) Use Inequality mode to solve for x: $5x + 6 \geq x^2$.

3. (a) Solve the linear system: $x + 2y = 8$
   $3x - y = -11$

   (b) Solve the linear system: $2a + b = 13$
   $b - 2a = 1$

4. Solve the linear system: $2x + y + z = 3$
   $x = 4y + z$
   $z + 5 = x + y$

5. (a) Find the roots of the function $f(x) = x^3 - 2x - 1$.
   (b) Use the answer from part (a) to find the solutions of $x^3 - 1 = 2x$.
   (c) Solve $x^3 < 2x + 1$

6. (a) Make a table of values for $f(x) = x^3 - x^2 - 4$ between $x = -10$ and $x = 10$.
   (b) Use your table to find $f(7)$, the value of the function when $x = 7$.
   (c) Use your table to find the value of x when $f(x) = -16$.

7. Consider the function $f(x) = x^3 - 2x + 1$. There is a solution to the equation $x^3 - 2x + 1 = 0$ between $x = 0.6$ and $x = 0.7$. Use a table to find this value of x to two decimal places.

8. Solve $2^x = x + 2$.

9. Use a table of values to decide how many solutions the equation $x + 2^x = 5$ has. Then find a solution to four decimal places of accuracy in the interval $1 < x < 2$.

10. Solve $3x^2 + 5 = 0$, and explain why both of the roots are complex numbers.

11. Find the real solutions, if any, to the equation: $x^2 + 2x + 2 = 3 - x^4$.

12. Consider the following formula, which shows the population y of a country after B years with an annual population growth rate of x%, starting with a population of A.

$$y = A\left(1 + \frac{x}{100}\right)^B$$

   (a) Enter this equation into the calculator and use the Solver to find out what population growth rate would be needed for a country that has 60 million people today to reach a population of 90 million in ten years' time.

   (b) Edit the equation and use the Solver to find out how many years of steady population growth at 2% per annum will be required for a country of 45 million to reach a population of 70 million.

13. Solve (a) $5:11 = x:6$ (b) $\frac{13}{4} = \frac{5}{a}$

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. A cone has a volume given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

   (a) Calculate the volume if $r = 2.5$ cm and $h = 8$ cm.

   (b) Calculate the height if the volume is 30 cm^3 and the radius is 2.3 cm.

   (c) Calculate the radius if the volume is 24 cm^3 and the height is 10 cm.

2. Explain why there is no solution to the linear system:

$$\begin{aligned} 3x - y &= 7 \\ 6x - 14 &= 2y \end{aligned}$$

   Can you find some other linear systems for which there is also no solution? Check with your calculator.

3. A ball is thrown in the air from a platform. Its height in metres, x seconds after being thrown, is given by

$$h(x) = 5x + 7 - 4.9x^2$$

   (a) Use a table to find the height of the ball when $x = 0.5$, $x = 0.7$ and $x = 0.9$.

   (b) How high is the platform?

   (c) When does the ball hit the ground?

   (d) In which direction was the ball thrown? up or down?

4. Consider the equation $\sin x = 0.4$.

   (a) Find the solution(s) for x in the interval $0 \le x \le 360^{\circ}$.

   (b) Are there any other solutions to the equation? Investigate these.

5. Use the calculator to find solutions to the equation: $(x + 1)^2 - 2x = x^2 + 1$. Explain why there are so many solutions.

6. The following system of equations contains coefficients (based on physical measurements) that have been rounded correct to one decimal place:

$$\begin{aligned} 2.4x + 5.7y &= 4.2 \\ 3.4x + 8.3y &= 3.2 \end{aligned}$$

   So, for example, the coefficient of 2.4 in the first equation represents a number in the interval from 2.35 to 2.45.

   Investigate the effects on the solution of these equations of using rounded coefficients like those above. (For example, try 2.37 and 2.42 instead of 2.4: how do the solutions compare?)

   Comment on the practical implications of your observations.

   Systems of equations like this are described as *ill-conditioned*.

Notes for teachers

In this chapter, ways of using the *ClassWiz* to solve equations numerically are explored. Table mode can be used to numerically obtain approximations to roots and solutions. Equation/Function mode allows for polynomial equations and for systems of linear equations in up to four variables to be solved. Inequality mode addresses polynomial inequalities, while Ratio mode addresses proportions. The *Solve* facility allows for nonlinear equations to be solved and formulae used efficiently. The text of the module is intended to be read by students and will help them to see how the calculator can be used with various kinds of equations. While the exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to undertake the activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to Exercises

1. $x = 4.5$ 2. (a) -2.19, 3.19 (b) $-1 \leq x \leq 6$ 3. (a) $x = -2, y = 5$ (b) $a = 27/5, b = 11/5$
4. $x = 23/15, y = 1/5, z = -4/15$ 5. (a) -1, -0.618, 1.618 (b) $x = -1$, -0.618 or 1.618 (c) $x < -1$, $-0.618 < x < 1.618$, 6. (b) 290 (c) -2 7. Tabulate between 0.61 and 0.62, then between 0.618 and 0.619 to see solution of $x \approx 0.62$ to 2 decimal places. 8. -1.69, 2 9. Only one solution, $x \approx 1.7156$
10. $x \approx \pm 1.29\boldsymbol{i}$. Solutions are complex because the equation $3x^2 = -5$ has no real solutions, as $3x^2$ is always positive. 11. $x \approx -1.1841, 0.4047$ 12. (a) 4.14% (b) 28.3 years 13. (a) 30/11 (b) 20/13

Activities

1. Parts (b) and (c) can be solved through use of the *Solve* command, although students will need to write the formula with appropriate calculator variables for V, r and h. Notice in part (b) that this approach avoids the need for students to algebraically transpose the equation, which many find difficult. [Answers: (a) 52.36 (b) 5.42 (c) 1.51]

2. Encourage students to use Equation mode first to see for themselves that an error is given. The two equations are *dependent*, as the second involves doubling each coefficient, and so no further information is added by the second equation. If graphed, the two equations would produce only a single line. Students should be able to generate further examples in the same way, with one equation a multiple of another, and verify that they cannot be solved by the calculator.

3. This activity involves using a table to explore a practical situation. Although it might also be regarded as an activity exploring a function, answering most of the questions is equivalent to solving an equation. It might be a good practice to ask students to write the equations that they are effectively solving by evaluating the function. [Answers: (a) 8.275, 8.099, 7.531 (b) 7 m (c) when $h(x) = 0$, $x \approx 1.8097$ (d) up.]

4. The equation has an infinite number of solutions. Students should look for patterns in the solutions. They will need to adjust the starting values for x to see later values. A sketch graph of the two functions $f(x) = \sin x$ and $g(x) = 0.4$ will help them to see how and why the patterns arise. [Answers: 23.5781° and 156.4218°]

5. This activity is concerned with the idea of an identity, for which there is an infinite number of 'solutions', although it is not common to describe an identity as a kind of equation. Whichever value for X the students start with when using the *solve* command will be a 'solution' generated by the calculator. [Remind students that the = sign has a variety of meanings in mathematics.]

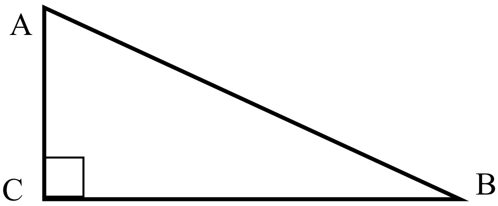
6. This activity is intended to encourage students to appreciate the need to think about equations, rather than mechanically solve them. Small changes in the coefficients lead to very large changes in the solutions. If these changes are the result of measurement errors, as suggested in the activity, caution is needed to interpret them. If a graphing mechanism is available, the two lines associated with the two equations can be seen to be almost parallel.

Module 5
Trigonometry

A calculator is a useful tool for many aspects of trigonometry, both for solving problems involving measurement and understanding relationships among angles. To start with check that your calculator is set to use degrees (by looking for a small D symbol on the display). Use SET UP (SHIFT MENU) to change it if necessary. Start this module in Calculation mode, by tapping MENU 1.

Trigonometry and right triangles

Definitions of trigonometric functions can be based on right triangles, such as the one shown below, for which C is a right angle.



For this triangle, sine B = $\dfrac{AC}{AB}$, cosine B = $\dfrac{BC}{AB}$ and tangent B = $\dfrac{AC}{BC}$.

In addition, because the triangle is right angled, the Theorem of Pythagoras allows us to see the relationship between the lengths of the two sides (AC and BC) and the hypotenuse (AB):

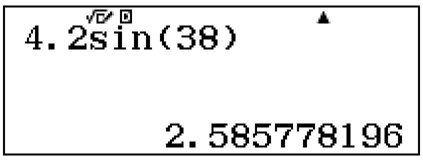
$$AC^2 + BC^2 = AB^2$$

Together, these relationships allow us to determine all the sides and the angles of a right triangle, even when only some of the information is known.

For example, if we know that angle B is 38^o and that AB = 4.2 m, we can find the length of AC using the definition of sine above:

$$\sin B = \frac{AC}{AB} = \frac{AC}{4.2}, \text{ so } AC = 4.2 \sin B.$$

Computations like this are easily performed on the calculator. It is a good practice to include the bracket after the angle size, even though the calculator will calculate correctly without it.



Notice that it is not necessary to use a multiplication sign in this case to find that AC ≈ 2.59 m.

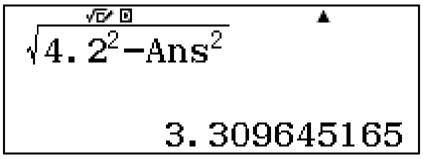
The screen above shows that the calculator gives the result to many places of decimals, but care is needed in deciding the level of accuracy of calculations like this. Since the original measurement of AB, which is used in the calculation, is given to only one place of decimals, it would be consistent with this to give AC to only one place of decimals too: AC ≈ 2.6 m.

To find the length of the other side, BC, a similar process could be employed, using the cosine of B. Alternatively, the Pythagorean Theorem can be used to see that

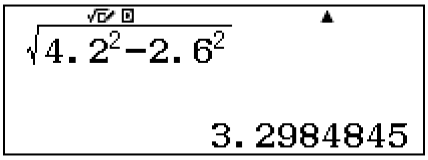
$$BC^2 = AB^2 - AC^2$$

$$\text{and so } BC = \sqrt{AB^2 - AC^2}$$

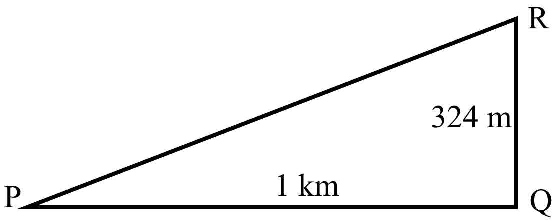
This can be readily determined from the above calculator result (which shows AC) as follows:



So BC ≈ 3.3 m. Notice that it has not been necessary to record any intermediate results here, and nor has it been necessary to use approximations (such as AC ≈ 2.6), which is likely to introduce small errors. In general, it is better to not round results until the final step in any calculations on calculators or computers. In this case, use of an approximation for AC results in a (slightly) different and slightly less accurate result for BC:



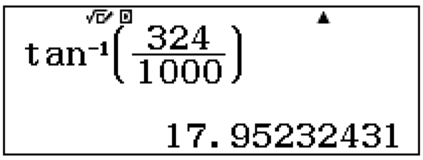
The calculator can also be used to determine angles in a right triangle, using inverse trigonometric relationships. For example the inverse tangent of an angle can be accessed with SHIFT tan ($\tan^{-1}$). If you know a perpendicular height and a distance, you can use this to find an angle of elevation. In the diagram below. QR represents the height of the Eiffel Tower in Paris. The tower is 324 metres high. P is a point one kilometre away from the base, measured on level ground. What is the angle of elevation from P to the top of the tower?

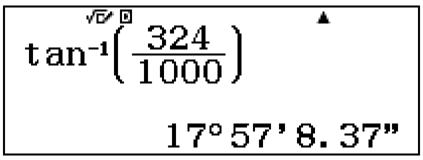


$$\text{Since } \tan P = \frac{RQ}{PQ},$$

$$\text{Angle P} = \tan^{-1}\frac{RQ}{PQ}.$$

The calculator gives the result using decimal degrees, as shown below on the left.

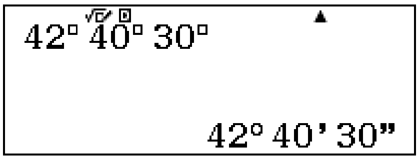


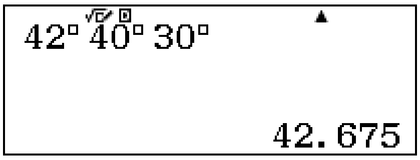


To represent this angle using degrees, minutes and seconds, tap the °'" key, to get the result shown at right above. Once again, care is needed to express results to a defensible accuracy. In this case, one kilometre from the Eiffel Tower, the angle of elevation from the ground is about 18°.

When using the calculator for trigonometry, it is sometimes necessary to enter angles in degrees, minutes and seconds. However, the calculator works with decimal degrees. To see the relationship between these two ways of using sexagesimal measures, enter an angle in degrees, minutes (and possibly seconds also) and tap [=] to see the decimal result.

To enter the angle, tap the [°’”] key after each of the degrees, minutes and seconds are entered; although a degree symbol is shown each time, the calculator interprets the angle correctly, as the first screen shows after tapping [=]. Tap the [°’”] key to see the angle in decimal degrees.

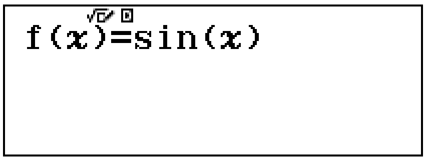


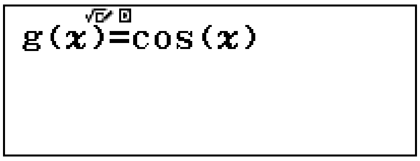


Tap the [°’”] key again to see the angle in degrees, minutes and seconds.

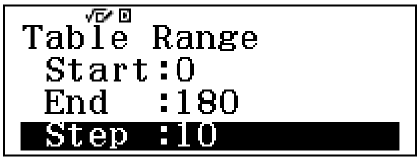
Tables of values

It is instructive to examine tables of values for trigonometric functions, in order to see how the functions depend on the angle. To do this use [MENU] [9] to select Table mode and enter the functions concerned, using x as the variable. (Use the x key.) In the screens below, we consider the sine and cosine functions. Tap [=] after entering each function.

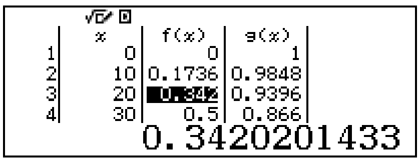


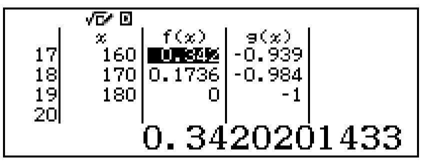


As only 30 values are permitted, a good choice for the table is to start at $x = 0^o$ and end at $x = 180^o$, with a step of 10^o. (Degree symbols are not used, however.) Tap [=] after each value is entered.



Use the cursor keys to scroll the table, showing the values of the sine and cosine of the angle in the final two columns. Notice how the values of sine increase up to a maximum of 1 from 0^o up to 90^o, and then decrease to 0 again at 180^o. The cosine values also show a pattern, but it is a different one. These tables will allow you to quickly sketch graphs of the two functions, which will also help to illustrate the relationships involved.

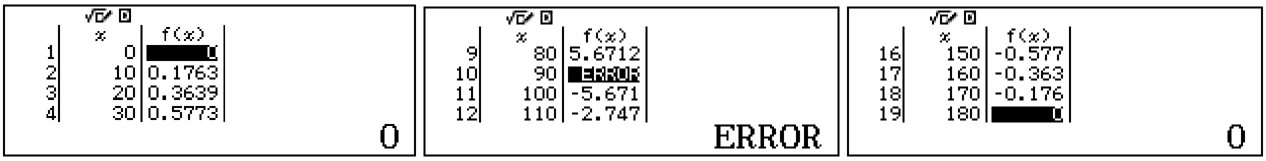




Check also that angles the same distance from 0^o and 180^o have the same sine value, as illustrated above for 20^o and 160^o.This illustrates the general relationship that $\sin x = \sin (180^o - x)$. Again, the relationship is different for cosines, with $\cos x = - \cos (180^o - x)$.

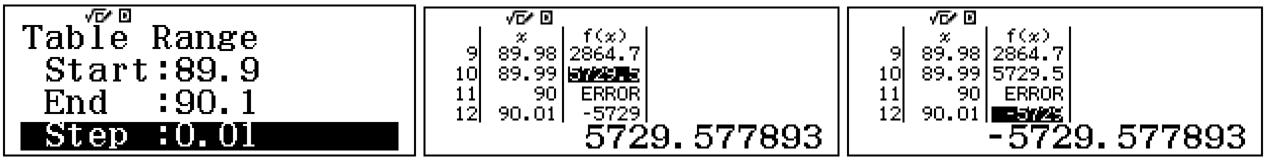
You can also compute values for $\sin x$ and $\cos x$ for $180^o \le x \le 360^o$ and observe other relationships. Remember that the calculator is restricted to 30 rows in a table, however, so a different choice of table step (such as 15^o) is needed.

While the sine and cosine functions have a special relationship with each other, the tangent function follows a different pattern. To explore this, firstly construct a table of values for $f(x) = \tan x$ from 0^o to 180^o in steps of 10^o. (Merely tap [=] instead of defining a second function, to construct a table of just one function.) Parts of this table are shown below.

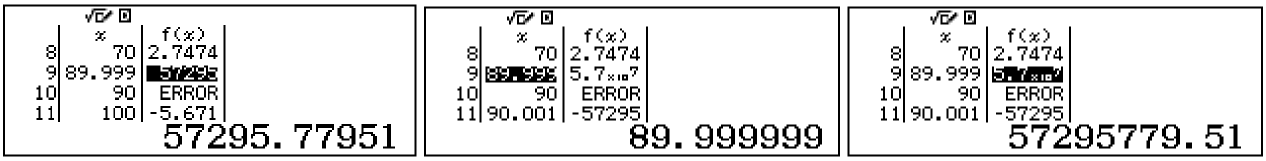


As there is no value defined for tan 90^o, the calculator shows an error at that point.

To examine more closely the behaviour of the function near 90^o, tabulate values of the function that are close to 90^o. There are two ways of doing this. In the screens below, the step has been changed to 0.01, and the table range changed to ensure no more than 30 values are obtained.



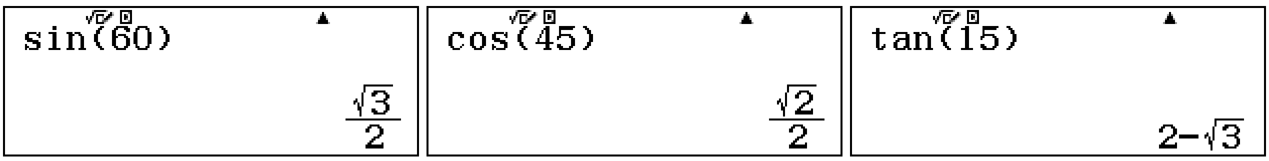
An alternative method is to change individual x values by highlighting them and entering a new value, followed by [=]. The tabulated values are changed accordingly. The screens below show some examples of this, starting with the original table. Notice from the last two screens that values exceeding the table's display width can both be entered and displayed.



You can see from these tables that there is a discontinuity at $x = 90^o$, around which the tangent function changes sign, from a large positive value to a large negative value. These tables will also help you to visualise the shape of a graph of the tangent function from 0^o to 180^o.

Exact values

Although practical measurement always involves approximations (as no measurement is ever exact), it is also interesting to study the exact values of some trigonometric relationships. You may have noticed that the calculator provides some of these, three of which are shown below.

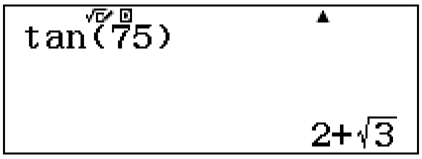


It is often possible to see the origins of these values, by drawing suitable triangles. For example, an isosceles right triangle includes angles of 45^o, while an equilateral triangle has angles of 60^o.

In addition, you can use these and other exact values to find the exact values of the sine, cosine and tangent of other angles, using various formulae for combinations of angles. To illustrate, consider the formula for the tangent of a sum of two angles:

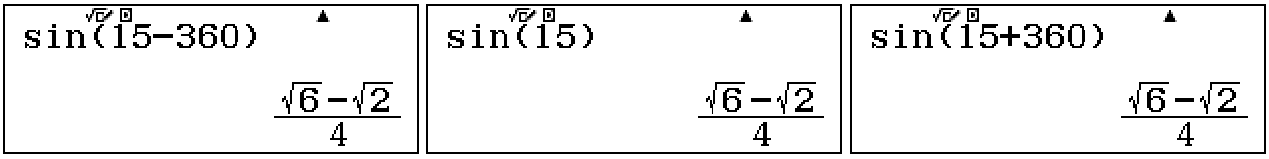
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

You can use the exact values of tan 45° = 1 and tan 30° = 1/√3 to calculate for yourself tan 75°. You should be able to obtain the same exact result as the calculator, shown below.



Experiment in this way with other addition and subtraction trigonometric formulae for sine, cosine and tangent.

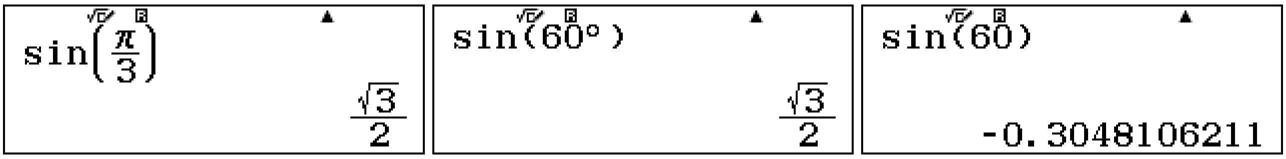
The calculator will also help you to experiment to see how trigonometric values repeat. For example, the screens below show how values for the sine function repeat every 360°.



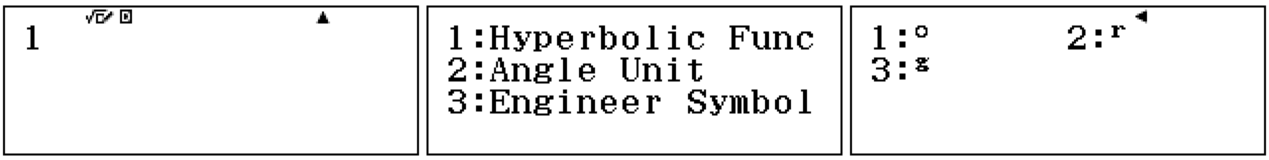
Radian measure

Radian measure provides a different way of measuring angles from the use of degrees, by measuring along a circle. An angle has a measure of 1 radian if it cuts an arc of a circle that is one radius long. As there are 2π radius measures in the circumference of a circle, the measure of a full rotation in a circle is 2π (radians), the same angle as 360° in degrees (or sexagesimal measure.)

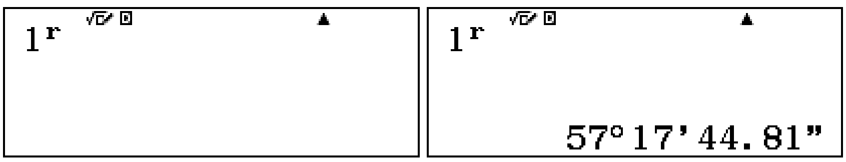
If you are using radian measures often, it is best to use SET UP to convert the angle measures to radians. Once this is done, the small R in the display will remind you of the setting. Notice in the screens below that this has been done, so that the first evaluation is correct. In the second evaluation, for sin 60°, the degree symbol has been added (using the OPTN 2 Angle Unit menu) to override the radian setting temporarily. The third screen shows that the calculator will assume the measure is in radians when it is set to radians, and shows the sine of 60 radians.



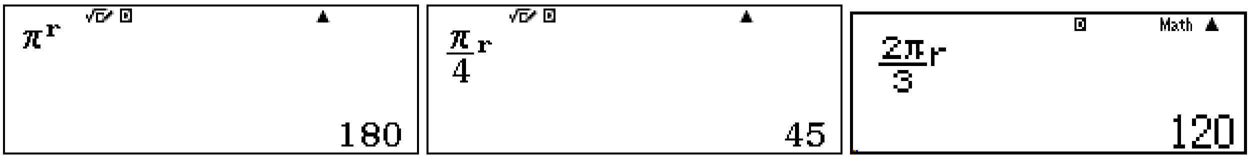
You can convert an angle measure from radians to degrees when the calculator is set to degrees. Start by entering the angle size. Using the OPTN menu, tap 2 to choose an angle unit and then 2 again to indicate that the measurement is in radians. The conversion of 1 radian to degrees using this process is shown below.



Tap = to finish the conversion. Notice below that °'" has been used to complete the conversion from decimal degrees to degrees, minutes and seconds. It is clear that one radian is a little over 57°.

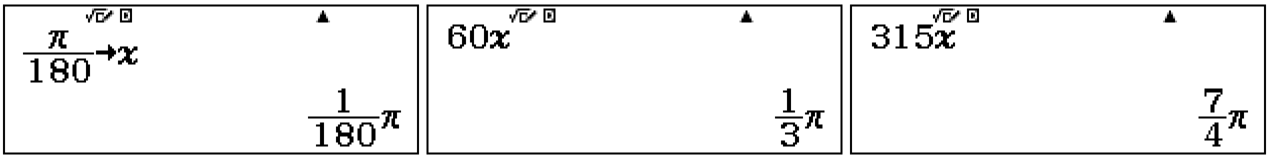


Other important values are shown below, using the SHIFT $\times 10^x$ (π) key.

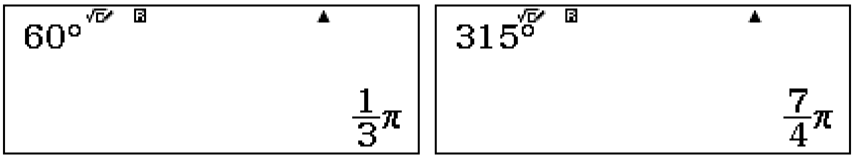


Conversions from degrees to radians can be made directly by multiplying by $\frac{\pi}{180}$.

You could do this conversion by storing the multiplier into a variable memory (using STO) and then multiplying by the variable whenever conversion was needed. The screen below shows this process. (Of course, you need to remember that the multiplier has been stored in memory x.)



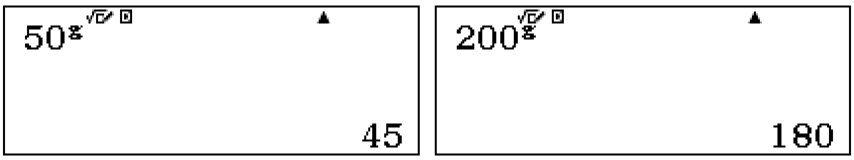
If the calculator has already been set to radian mode in SETUP, however, then a similar process can be used as above: enter the angle and use the OPTN menu to indicate that it is measured in degrees Two examples are shown below. (Notice the small R in the display, indicating radian mode.)



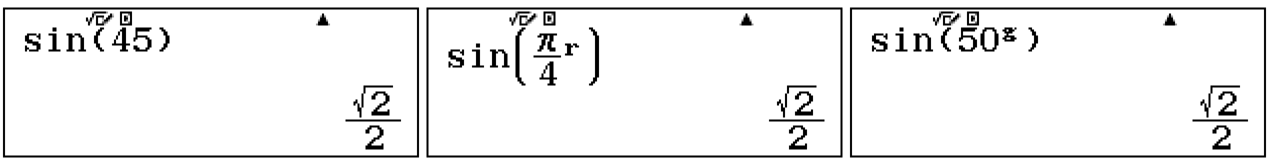
Gradian measure

The idea of gradian measure was invented to provide a way of measuring angles that would be consistent with the metric system, which relies on powers and multiples of 10 for both measures and numbers. It is not widely used today, except for some surveying purposes. There are 100 gradians in a right angle and so there are 400 gradians in a full circle. You can use similar processes as above to convert between measures.

For example, in the screens below, note that the calculator is set in degrees (see the small D symbol). The equivalents to some measures in gradians are shown.



The same values and relationships apply for trigonometric functions, regardless of the way an angle is measured, as the following example demonstrates. In each case, the sine of the same angle is being determined, even though different measurements of the angle are used.

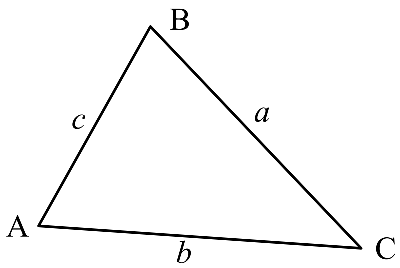


Solving triangles with the Sine Rule

At the start of this module, we described the use of trigonometry with right triangles. However, most triangles are not right triangles, so it is very helpful to use trigonometry with other triangles. The Sine Rule connects the lengths of the sides of any triangle with the sines of the opposite angles. The triangle does not need to be right-angled for the rule to be used. Radians can be used too.

For any triangle ABC with sides a, b and c units in length opposite the angles A, B and C respectively, the Sine Rule is:

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

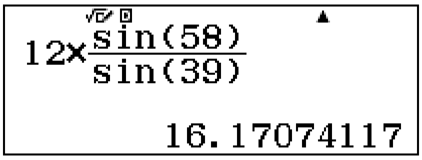


The Sine Rule can be used to solve problems involving triangles given (a) two angles and one side or (b) two sides and a non-included angle. Here are examples of each of these two cases:

(a) If AB = 12 cm, C = 39° and B = 58°, you can find the length of AC from:

$$\frac{b}{\sin 58^{\circ}}=\frac{12}{\sin 39^{\circ}} \text{ and so } b=\frac{12\sin 58^{\circ}}{\sin 39^{\circ}}.$$

The calculator allows you to complete this calculation in a single step:

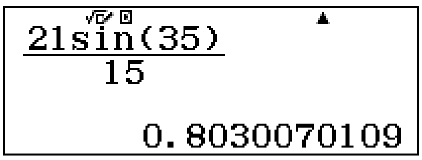


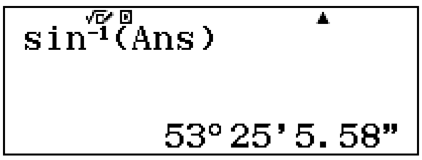
So, b = 16.2 cm, correct to one decimal place.

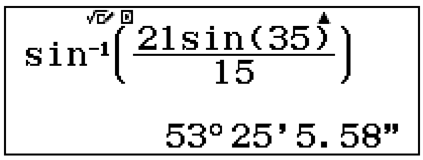
(b) If AC = 21 cm, AB = 15 cm and C = 35°, you can find the size of angle B from:

$$\frac{\sin B}{21}=\frac{\sin 35^{\circ}}{15} \text{ and so } \sin B=\frac{21\sin 35^{o}}{15} \text{ and hence } B=\sin^{-1}\left(\frac{21\sin 35^{0}}{15}\right)$$

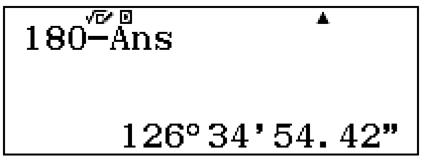
You can use the calculator to complete the calculation in two steps, or in a single step (shown in the third screen below):





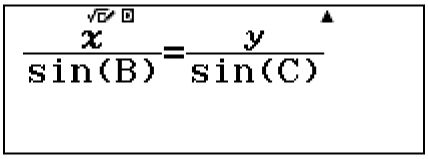


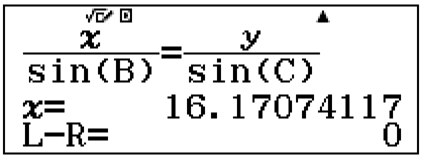
However, caution is needed here! There are two possible solutions for angle B. There are two different angles with a sine of 0.8030, as an angle and its supplement have the same sine (as noted earlier in this Module). The other possibility is 180° – 53.42°. To calculate this efficiently, the angle obtained can be subtracted from 180°, by using the Ans key immediately, as shown below:



So this particular triangle could have angles 35°, 53.42° and 91.58° or it could have angles 35°, 126.58° and 18.42°. Draw a diagram to see this for yourself. Without further information about the triangle, there are two possible solutions in this case. This situation is described as the *ambiguous case*. Sometimes there is enough information provided to reject one of the possibilities, and at other times, the Sine Rule will provide two solutions for a triangle.

You can use the *Solver* (described in Module 4) efficiently to solve equations directly involving the Sine Rule, but need to use different variables to represent the side lengths (as the calculator has a limited number of variables available). In the screen below, we have used x for the length of side b and y for the length of side c.





Consider example (a) above again. After tapping SHIFT CALC to start the solver, you will need to enter values for B, y and C and then solve for x. The solution of $x \approx 16.2$ cm (to one decimal place) is shown.

Solving triangles with the Cosine Rule

The Cosine Rule is another rule that is very useful for finding angles and side lengths in triangles that are not right angles. If a, b and c are the lengths of the three sides of triangle ABC, then

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Rearranged to find an angle, the Cosine Rule can be written as:

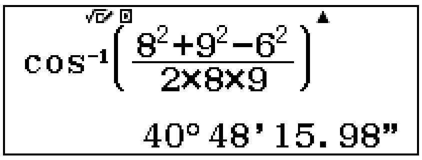
$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right).$$

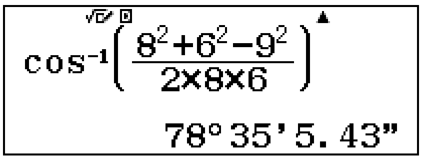
The first relationship can be used efficiently in the calculator to find the length of one side of a triangle when the lengths of the other two other sides and the size of their included angle is known. Alternatively, if the lengths of all three sides of a triangle are known, the sizes of the angles can be determined efficiently using the second formula.

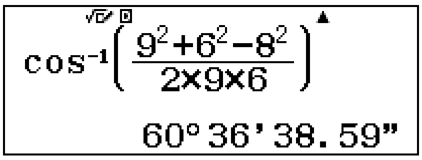
For example, consider a triangle with side lengths 6, 8 and 9. The size of the angle opposite the shortest side (i.e. the side of length 6) can be determined by the second version of the Cosine Rule:

$$A = \cos^{-1}\left(\frac{8^2 + 9^2 - 6^2}{2 \times 8 \times 9}\right)$$

This can be determined with a single calculation on the calculator, as the first screen below shows.



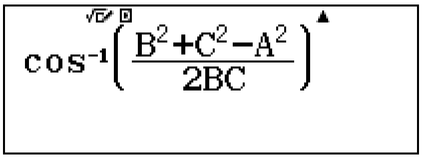


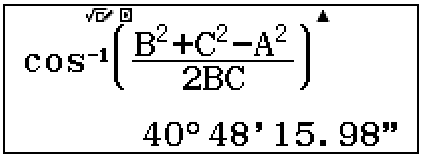


The other two angles can then be found by editing this expression (using ◀), as shown above:

The third angle could also be determined by subtracting the sum of the first two angles from 180°, although this is probably a little more difficult than editing the expression. Again, sensible accuracy needs to be considered here, bearing in mind that the triangle sides are given to the nearest whole number. Perhaps the angles might be given as approximately 41°, 78.5° and 60.5° respectively.

The version of the Cosine Rule used here can also be efficiently evaluated in the calculator using CALC. The screen below shows the formula for finding angle *A*. After tapping CALC and entering values for *A*, *B* and *C*, the result is obtained as shown. Note that the °''' key was also tapped.

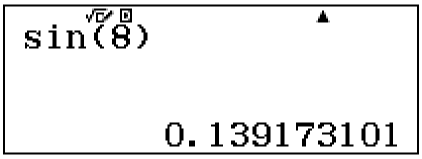


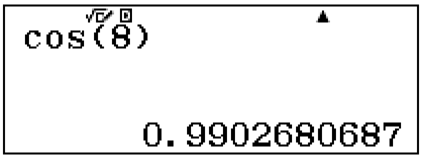


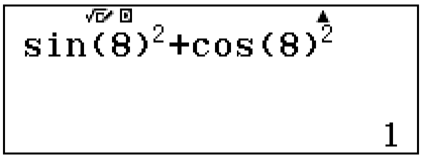
The Pythagorean Identity

There is an important relationship between the sines and cosines of angles. Called the *Pythagorean Identity*, the relationship is usually expressed as: $\sin^2 A + \cos^2 A = 1$

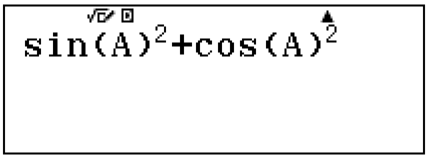
For any angle *A*, the square of the sine and the square of the cosine add to 1. Be careful with the notation of $\sin^2 A$, which means $(\sin A)^2$. To illustrate the Pythagorean Identity for an angle of 8°, notice in the third screen below that, because the calculator will not permit the x^2 key to be tapped immediately after the sin key, it must be tapped after the closing parenthesis for each term.

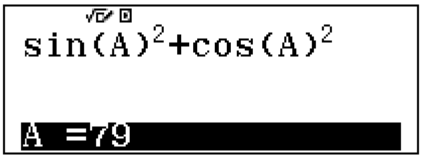


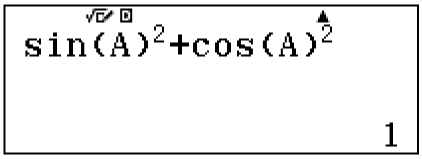




To see that the identity is always true (which is a requirement for a statement to be described as an identity), enter $\sin^2 A + \cos^2 A$ into the calculator and use CALC several times to check some values for *A*, as shown below for $A = 79°$.

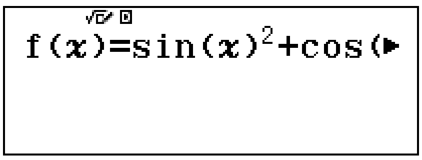


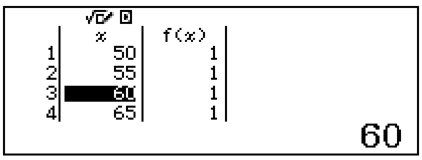


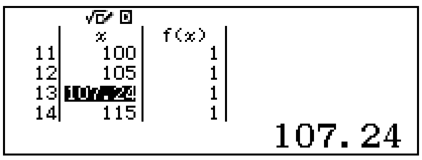


Check a range of values (positive, negative, large and small) and check what happens if a different angle measure is used instead of degrees. You will see that this powerful identity describes a relationship that is *always* true.

Another way to see this is to construct a table of values, showing that $f(x) = \sin^2 x + \cos^2 x$ always has a value of 1, regardless of the value of *x*:







Importantly, the Pythagorean relationship allows you to determine one ratio for an angle, if you know another one. For example,

$$\cos A = \sqrt{1 - \sin^2 A} \quad \text{and} \quad \tan A = \frac{\sin A}{\cos A} = \frac{\sqrt{1 - \cos^2 A}}{\cos A}.$$

Check that these relationships are correct by evaluating them on the calculator for some angles.

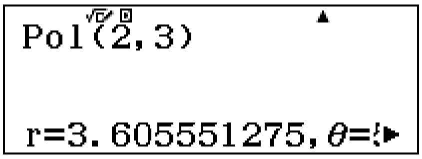
Coordinate systems

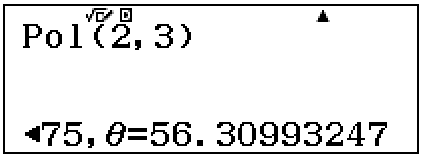
As well as dealing with triangles, trigonometry is also involved in representing points on the plane. *Rectangular coordinates* are the most common way of doing this, specifying how far a point is to the right of the vertical axis and how far it is above the horizontal axis. So, the point (2,3) is 2 units to the right and 3 units up.

In general, rectangular coordinates are represented by (x,y). Rectangular coordinates are often described as *Cartesian coordinates*, named after their inventor, mathematician and philosopher René Descartes.

An alternative system uses *Polar coordinates*, measuring the direct distance of the point from the origin and the angle of rotation in an anticlockwise direction from the horizontal. Polar coordinates are represented by (r, θ).

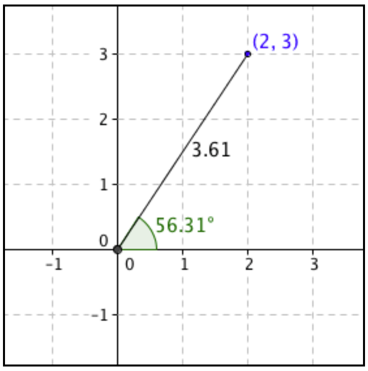
The *ClassWiz* includes commands SHIFT + (Pol) and SHIFT − (Rec) to convert between these two systems. The screen below shows how to do this to convert rectangular coordinates to polar coordinates. Make sure you are using the correct angular units (degrees or radians). You need to use a comma, available via SHIFT), in order to enter the command.

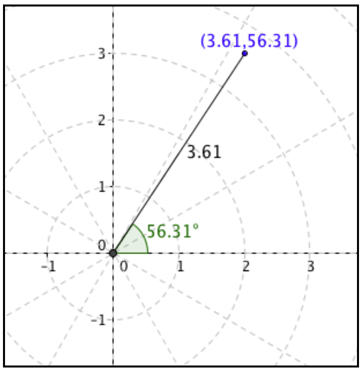




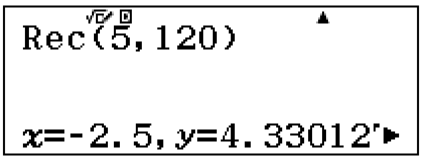
Because the numbers are too long to fit on the screen, use the ▶ key to see the entire output. In this case, the point (2,3) can be represented in polar coordinates approximately as (3.61, 56.31), indicating that it is 3.61 units from the origin and 56.31° anticlockwise from the horizontal.

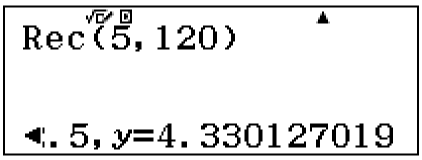
Check in the diagrams below that the same point can be accurately described using either Cartesian or polar coordinates.





In a similar way, the screens below show the rectangular coordinates of the point that is 5 units from the origin and rotated 120° counter clockwise.



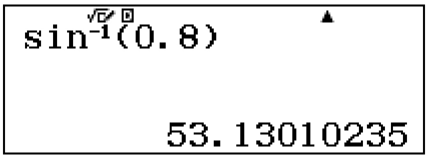


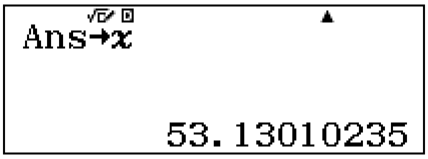
So the point (5,120°) can be approximately represented in Cartesian coordinates as (-2.50,4.33). Notice that this point is in the second quadrant, as the x-value is negative and the y-value is positive.

Trigonometric equations

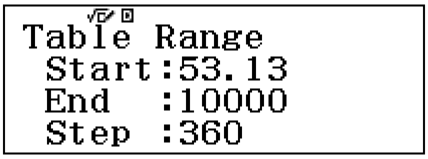
Care is needed when solving trigonometric equations, as there are often many solutions, unless the domain is restricted. Consider, for example, the very simple equation: $5\sin x - 1 = 3$.

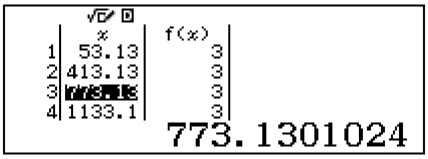
If the equation is rearranged to $\sin x = 0.8$, a solution is $x = \sin^{-1}0.8$, so that $x \approx 53.13^{\circ}$:

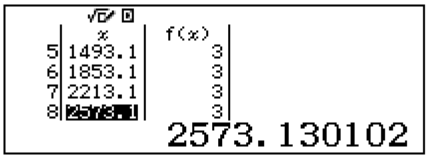




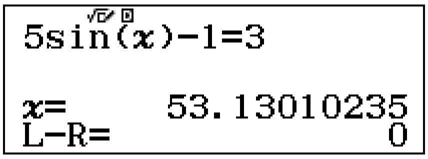
For $0^{\circ} \le x \le 90^{\circ}$, $x \approx 53.13^{\circ}$ is the only solution to the equation. However, there are other values of x that are solutions, since the sine function is periodic, with values repeating every 360°. So other solutions are $360^{\circ} + 53.13^{\circ}$, $720^{\circ} + 53.13^{\circ}$, … or, in general, $360k + 53.13^{\circ}$ (correct to 2 decimal places). You can see these solutions in a table for $f(x) = 5\sin x - 1$, starting with 53.13 and increasing in steps of 360. (It is best to store the starting value into a memory (such as x above) and then use that variable in the *Start* of the table, to efficiently get the correct value.). Extracts from the table are shown below, indicating that there are many solutions (in fact, an infinite number), i.e., values of x for which $5\sin x - 1 = 3$.

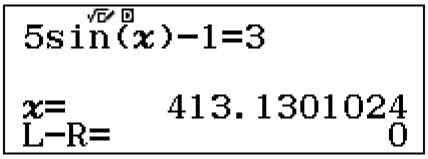


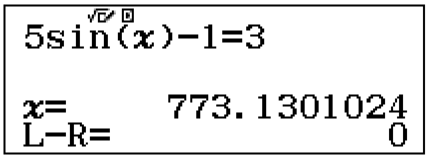




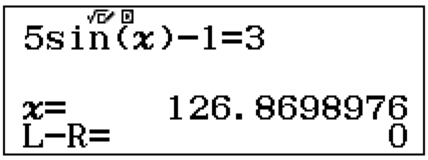
The *ClassWiz* provides the smallest solution to $x = \sin^{-1}0.8$, but if you solve the equation in the *Solver*, then other solutions will be provided by starting with different values for x. The three solutions below were generated with starting values of 0, 360 and 720 respectively.

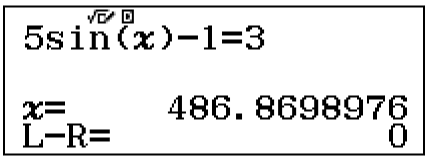


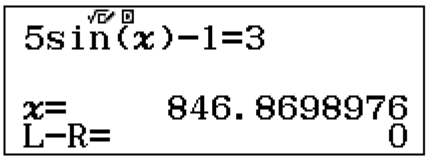




In this case, there are still more solutions, because of the supplementary angles property of the sine function: $\sin(180^{\circ} - x) = \sin x$. The solutions below were generated with starting values of 180, 540, 900, … .







If you consider the patterns here closely, you can see that, in general, there is a second set of solutions of the form $360k + 126.87^{\circ}$, for k an integer.

As before a table will show you that there are many values of x solving the equation.

In this case, if the equation includes a domain for x, such as :

$$5\sin x - 1 = 3, \text{ for } 0^{\circ} \le x \le 180^{\circ}$$

then there are only two solutions: $x = 53.13^{\circ}$, 126.87° (correct to two decimal places). But if no domain is specified, there is an infinite number of solutions: $x = 53.13^{\circ} + k360^{\circ}$, $x = 126.87^{\circ} + m360^{\circ}$, for k and m integers. (Note that k and m can be negative as well as positive.)

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. Find (a) sin 36^o (b) cos 2 radians (c) tan 70 gradians.

2. Convert 34.458 degrees to degrees, minutes and seconds.

3. The tangent of an angle in a right triangle is 0.6. What is the size of the angle?

4. Make a table of values for the cosine function for angles between 0 and 180 degrees. Use the table to decide how cos x and cos $(180^o - x)$ are related.

5. Give the exact value of sin 15^o.

6. An angle has size 1.4 radians. Give its size in degrees, minutes and seconds.

7. What is the radian measure associated with an angle of 31^o?

8. Give the polar coordinates associated with the point (6,5).

9. Give the Cartesian coordinates associated with a point with polar coordinates $(4, 240^o)$. In which quadrant is the point located?

10. Students want to calculate the height of a tree that is 50 m away from them. They measure the angle of elevation of the tree to be 17^o. Calculate the approximate height of the tree.

11. In triangle ABC, AC = 17.2 cm, $B = 35^o$ and $A = 82^o$. Find the length of BC, correct to one decimal place.

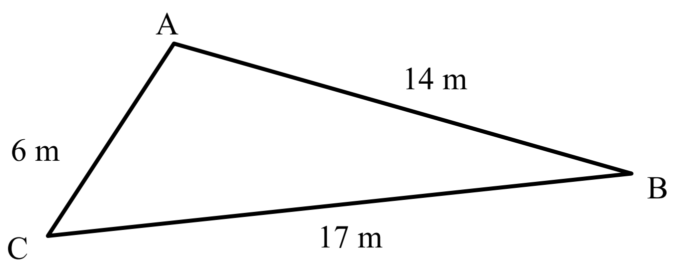
12. In triangle KLM, $K = 56^o$, KM = 13.5 m and LM = 16.8 m.

    (a) Use this information and the Sine Rule to find the remaining angles in the triangle.

    (b) Explain why the ambiguous case does not apply to this triangle.

13. Triangle ABC below is not drawn to scale.

    Use the side lengths and the Cosine Rule to find the sizes of the three angles to the nearest degree.



14. Solve $4 \cos x + 6 = 7$ for $0^o < x < 360^o$.

15. Solve $5 \sin x - 2 = 1$.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. Consider the right triangle shown on the first page of this module. Actual measurements in practice with measuring instruments always contain errors. Suppose the length of AB is regarded as 4.2 ± 0.1 metres and the size of angle B as $38^o \pm 5^o$, because of the tools used to measure. Explore the effects on the calculation of AC of these uncertainties of measuring.

2. Make a table of values for the sine and cosine functions, with angles (in degrees) from 0 to 360. Use a step of 15, so that the table will not be too large for the calculator. Study the table values carefully to look for relationships among the sines and cosines of angles in different quadrants. It may be helpful to write down the values on paper to make it easier to compare them.

   Look also to find connections between the sines and the cosines of angles.

3. Set your calculator to use radians instead of degrees. Construct a table of values for the sine function $f(x) = \sin x$ and cosine function $g(x) = cos(x)$, for values of x between 0 and π. Use a step of π÷24, using the π key on your calculator (i.e. SHIFT ×10ˣ) where necessary.

   (a) Compare your table with the one suggested in the text, which uses degrees. What symmetries do you see in the table?

   (b) Use a diagram of a unit circle to understand the reasons for the symmetry.

4. Set your calculator to degrees. Explore the relationships between angle sizes and sines.

   (a) For example, if an angle is doubled in size, does its sine also double? In other words, does $\sin 2A = 2\sin A$? Work with a partner, with one person finding $\sin A$ and the other finding $\sin 2A$ for various values of A. A table may also help.

   (b) In fact, $\sin 2A = 2 \sin A \cos A$. Test out this relationship using different values for angle A. Work with a partner, with one person choosing a value for A and each person evaluating one of the two sides of the equality. [Two good ways to evaluate an expression like $2 \sin A \cos A$ are to use the CALC command or to use a table.]

5. Draw a large triangle on paper and measure carefully the lengths of the three sides. You should be able to measure at least to the nearest millimetre. Then use the cosine rule to find the sizes of the three angles of your triangle to a suitable level of precision. After you have finished the calculations, check the angle sizes of the triangle with a protractor. How close are your calculations and your measurements? Try another example and compare with other students.

6. You might have noticed that some trigonometric values are given exactly by the calculator (using square root signs), while others are approximated with decimals. For example, ratios for angles of 30^o, 45^o, 60^o and 90^o are all given exactly; these can be obtained from the geometry of special triangles. In other cases, trigonometric formulas can be used to find values exactly. For example, here are two formulae used to find cosines for differences and halves of angles:

   $$\cos(A - B) = \cos A \cos B - \sin A \sin B \qquad \cos\frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}$$

   Experiment with these formulae to find exact cosines for other angles, and use the calculator to check your results.

Notes for teachers

This module highlights some ways in which the CASIO *ClassWiz* can support students to think about several aspects of trigonometry. All three angle measures (degrees, radians and gradians) are included, as well as conversions between rectangular (Cartesian) coordinates and polar coordinates. The calculator is a convenient tool for trigonometric calculations, such as those using the Sine and Cosine rules, and equation solving. The text of the module is intended to be read by students and will help them to see how the calculator can be used in various ways. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. (a) 0.5878 (b) -0.4161 (c) 1.9626 2. $34^{o}27'28.8''$ 3. 30.9638 degrees 4. Use a step of 10 degrees to see that cos $(180^{o} – x)$ = -cos x 5. (√6 - √4)/4 6. $80^{o}12'50.73''$ 7. 31π/180 = 0.5411^{R} 8. $(7.8102, 39.8056^{o})$ 9. (-2,-3.4641), in third quadrant. 10. 15.3 m 11. 29.7 m 12. (a) $L \approx 41.77^{o}$, M ≈ 82.23^{o} (b) If $L \approx 138.23^{o}$, sum of angles of triangle KLM would add to more than 180^{o}. 13. A ≈ 110^{o}, B ≈ 19^{o}, C ≈ 51^{o} . 14. 75.5^{o}, 284.5^{o} 15. $36.9^{o} + k360^{o}$, $143.1^{o} + m360^{o}$, *k, m* integers

Activities

1. This activity draws attention to the inevitability of errors in measurements and studies some likely consequences. In the example given, the length of AC could be as low as 2.23 m and as high as 2.93 m, a substantial range. We hope that this kind of exploration will discourage students from routinely using all the decimal places provided by calculator answers. It may be productive to discuss likely measurement errors with students, as a result of their attempts to measure lengths and angles in practice (noticing that the errors suggested for angles are greater than those for lengths).

2. This activity continues that suggested in the third page of the module, and includes an extension to angles that are not associated with triangles, such as those between 180^{o} and 360^{o}. Students might also be encouraged to consider relationships between the ratios of angles A and (A + 360^{o}). Activities of this kind are especially appropriate in association with drawings (such as a unit circle) and are intended to add meaning to formal relationships such as cos (180^{o} – A) = -cos A.

3. Relationships for angles measured in degrees and radians are the same, and this activity is intended to help students appreciate that changing the angle measure does not affect fundamental relationships. Notice that the calculator converts the exact values of π and π÷24 to decimals.

4. Many students seem to develop the misconception that sin $2A = 2 \sin A$, so this activity is intended to explore the situation numerically, to discourage such incorrect generalisations. It is a good idea for students to do this activity in pairs, as suggested, with each student having a calculator. Use of a pair of tables such as for $f(x) = \sin 2x$ and $g(x) = 2 \sin x$ is also effective.

5. Many practical activities can be undertaken with a calculator and actual measurements. This example will help students to see the practical value of the cosine rule, but also appreciate the significance of careful measurement, as it is inevitable that calculated and measured results will be not quite the same. Work of this kind may also discourage students from (over)interpreting excessive numbers of decimal places in calculator results.

6. This activity uses the calculator as a check on calculations derived more formally, and is quite demanding of students' manipulative skills. The calculator in fact gives exact results for angles that are a multiple of 15^{o}, such as $\sin(15^{o}) = \frac{\sqrt{6}-\sqrt{2}}{4}$, while the activity will help them to understand the origins of such relationships.

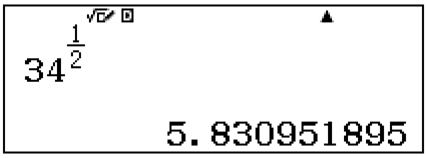
Module 6
Exponential and logarithmic functions

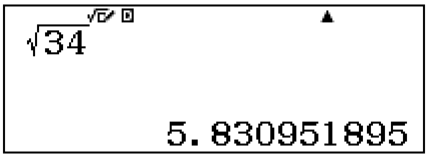
Exponential and logarithmic functions have an important place in mathematics and its applications. The CASIO *ClassWiz* calculator provides significant support for understanding and using these ideas. Start this module in Calculation mode, by tapping [MENU] [1].

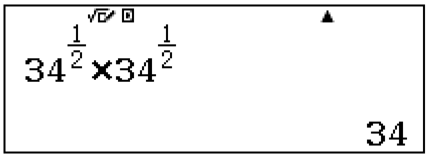
Exponents and roots

You are already familiar with powers of numbers, such as squares and cubes, and have used the calculator to evaluate these. The square and cube keys evaluate powers by repeated multiplication. That is, if you evaluate 34^2 on the calculator using the [x^2] key, it evaluates 34 x 34 to give the result. Similarly, the [x^{-1}] key uses division to evaluate reciprocals. Other exponents such as those involving fractions or decimals or higher powers require you to use the [$x^{■}$] key.

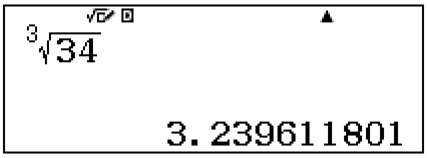
While the meaning of powers like 34^5 is clear, it is less clear what is meant by $34^{2/5}$ or $34^{0.43}$. We saw some special cases of fractional powers in Module 2, where raising numbers to the power of ½ is seen to be the same as finding a square root, as shown below.

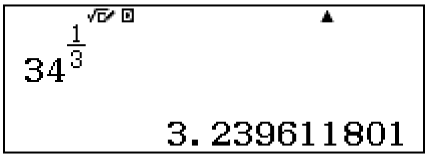


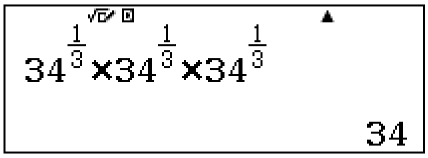




A similar sort of relationship is evident for cube roots:







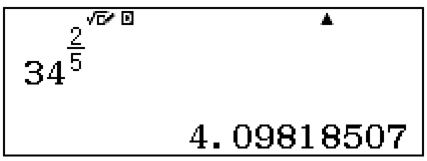
Properties like these depend on the laws of indices, since indices can be added when powers of the same base are multiplied, as shown below:

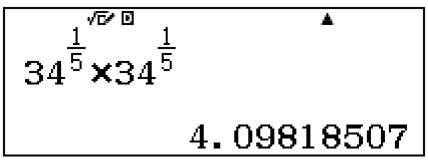
$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$$

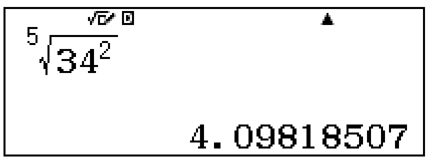
The same kind of thinking can help you to understand the meaning of numbers raised to fractional or decimal powers. Consider $34^{2/5}$, using the properties of indices:

$$34^{\frac{2}{5}} = 34^{\frac{1}{5}+\frac{1}{5}} = 34^{\frac{1}{5}} \times 34^{\frac{1}{5}} = \left(34^{\frac{1}{5}}\right)^2 \text{ or } 34^{\frac{2}{5}} = 34^{2\times\frac{1}{5}} = \left(34^2\right)^{\frac{1}{5}}$$

So, one way to think about it is as the square of the fifth root of 34, while another way is to regard it as the fifth root of 34 squared, as shown below:





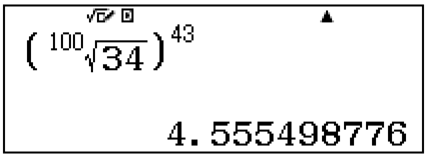


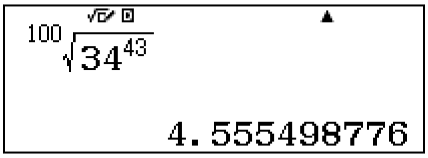
The calculator shows that these two ways of thinking about the number give the same decimal result. The exponent key $\boxed{x^{\blacksquare}}$ and the root key ($\sqrt[\blacksquare]{\square}$) are needed. Notice also that, this number is a little bit less than √34, shown above, which is to be expected as 2/5 is a little bit less than ½.

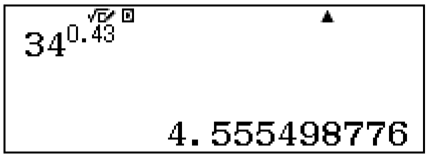
We have illustrated these properties for the number 34, but the same argument applies for any positive number. An important general relationship to help you understand the meaning of numbers raised to various powers is the following, for all integers *m* and *n* (≠ 0) and for any base $x > 0$:

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m \text{ or } x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}\ .$$

Although you can evaluate on the calculator a number raised to an exponent using either of the representations above, it is usually more convenient to evaluate it directly, using the $\boxed{x^{\blacksquare}}$ key. The screens below show all three possibilities for evaluating $34^{0.43}$.







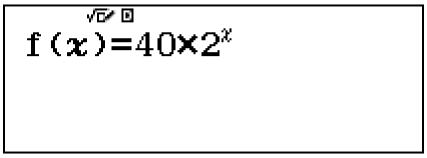
Notice again from the various screens above that, since 2/5 < 0.43 < ½ that the powers of a number have the same relationship: $34^{2/5} < 34^{0.43} < 34^{1/2}$.

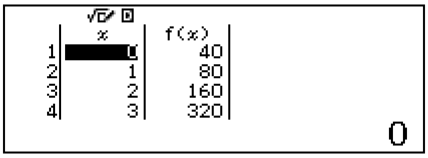
Exploring exponential functions

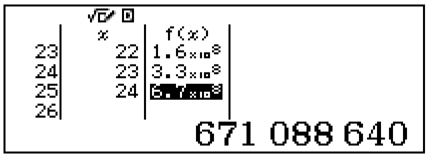
An *exponential* function is a function for which the exponent is variable. Many examples of exponential functions arise from situations of growth or decay. For example, consider a population of bacteria that doubles in size every hour, starting with 40. If *x* stands for the numbers of hours after the start, then the number of bacteria at any time after the start is given by the exponential function

$$f(x) = 40 \text{ x } 2^x\ .$$

A good way to understand how the bacteria population grows is to tabulate this function. Use Table mode, with $\boxed{\text{MENU}}$ $\boxed{9}$ and choose values for *x* from 0 (initially) to 24 (after one entire day).



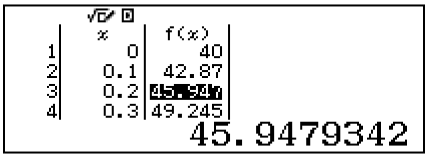


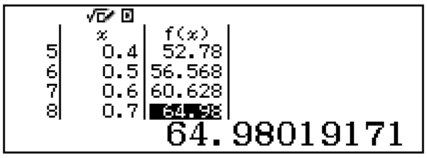


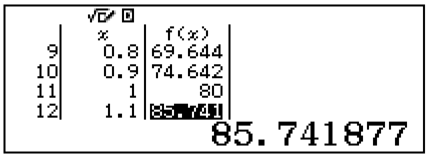
Scrolling the table shows that the bacteria population does start at 40 and doubles every hour. It is clear from the table that this kind of growth is very rapid. If you scroll to the bottom of the table, you will see that after 24 hours, the population is larger than 671 million bacteria!

Notice that when $x = 0$, $f(0) = 40 \text{ x } 2^0 = 40$, since a positive number raised to the power zero is always 1.

The growth can be examined over smaller intervals than hourly. For example, if you construct a table of values with a step of 0.1 instead of 1, you can examine how the bacteria population grows every six minutes (0.1 hours). (Reduce the maximum value in the table to 2.9, as only 30 tabulated values are permitted in the calculator.) Here are some extracts from the table:



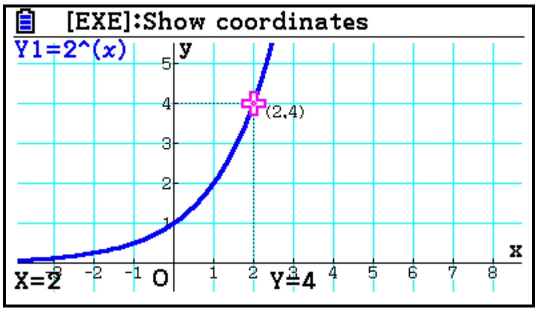




Notice that the exponential function used to model growth produces results that are not whole numbers (which is not sensible for bacteria). This is common for mathematical models, which always represent an ideal version of reality.

Notice also that in each successive period, the population grows a little more than previously. Between 0 and 0.2 hours, the population changed by (almost) 6, between 0.3 and 0.5 hours, it changed by more than 7, while between 0.6 and 0.8 hours, it changed by more than 9. This kind of change is typical of exponential growth. Explore some other parts of the table for yourself.

Graphs of exponential functions with a positive base larger than 1 have a distinctive shape. The graph below of $f(x) = 2^x$ was produced with a CASIO fx-CG20 graphics calculator, and shows a rapid growth for positive values of x. The function only has positive values, but they are quite small for $x < 0$. Notice that when $x = 0$, the value of the function is 1 (as $2^0 = 1$).



Some exponential functions do not involve exponential growth, but rather exponential *decay*. These are functions for which the base is between 0 and 1, so that higher exponents lead to smaller values of the function concerned. A good example of this is radioactive decay. Radioactive materials have a time called a *half-life*, after which half of the original material has disintegrated. If you know the half-life, you can determine the function involved.

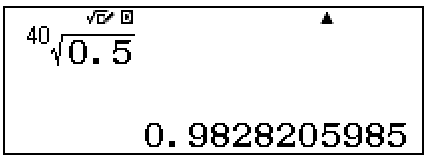
Suppose you have 20 g of a radioactive substance known to have a half-life of 40 days. Then a model for the amount of material left after x days is

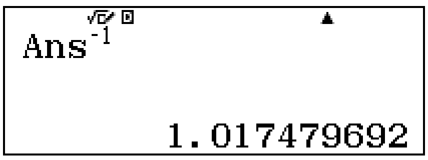
$$f(x) = 20b^x$$

Where b stands for the base of the model. If you substitute $x = 40$ (the half-life), then only 10 g is remaining:

$$10 = 20b^{40}$$

from which you can find $b = \sqrt[40]{0.5} \approx 0.98282 = 1.0175^{-1}$, as shown below:

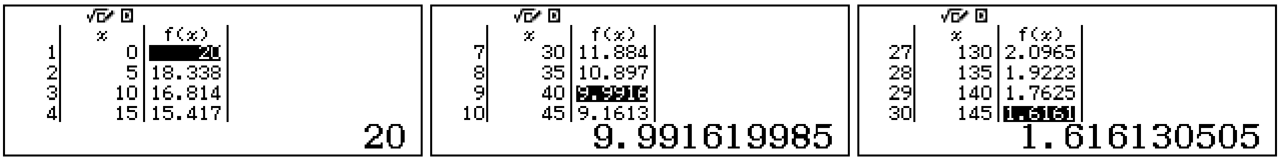




So this particular model for exponential decay can be expressed approximately as either

$$f(x) = 20 \times 0.9828^x \text{ or } f(x) = 20 \times 1.01748^{-x}$$

The first version has a base less than one, while the second has a negative exponent. If you construct a table of values for this function, using either of the two forms, you can see clearly that the value decreases as the variable (x) increases, and will eventually be close to zero as the material decays. Some values from a table are shown below to illustrate these properties:



Using exponential models

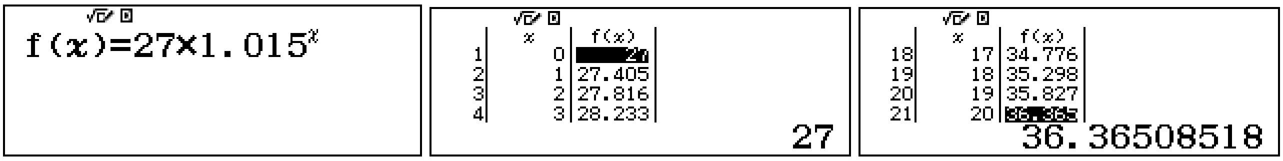
Exponential functions are very useful to model some real world phenomena. Exponential growth (or decay) occurs when a quantity increases (or decreases) by a constant factor in each time period. Many natural growth processes are of this kind, such as a growth of a country's population, which is usually described by an annual percentage rate. So, during each time period, if the percentage is constant, the actual growth is increasing because the population itself is increasing. Population changes are not as rapid as the bacteria growth, of course, but the growth is still of the same kind.

An example of this is Saudi Arabia, with a total population estimated to be about 27 million people in 2014 and an annual growth rate of approximately 1.5%. To use mathematics to model the population of Saudi Arabia, we will assume that the population growth rate is stable (although it is in fact likely to change over time).

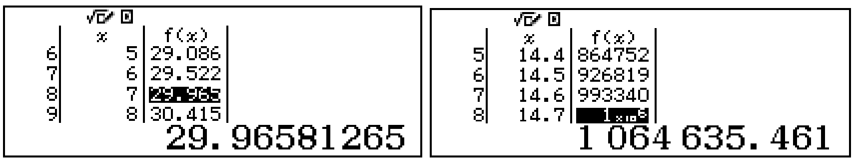
Each year the population increases by 1.5%. To recognise this as exponential growth you need to think of an additional 1.5% as the same as multiplying the population each year by 101.5% or 1.015. So the population x years after 2014 is given by the exponential function

$$f(x) = 27 \times 1.015^x.$$

A table of values will allow you to estimate the population after various years have passed. Some examples are shown below Notice that the population estimates are given in millions. So the model estimates that the population of Saudi Arabia will be about 36.4 million in 2034.



Exponential models like these are helpful to predict growth. They can be used to answer questions such as "When should we expect Saudi Arabia's population to reach 30 million, if the current rate of growth continues?" or, "After how many hours will the bacteria population be one million?" To answer questions of these kinds, we could use the CALC facility or use a table, choosing carefully suitable *Start*, *End* and *Step* values, as shown below.



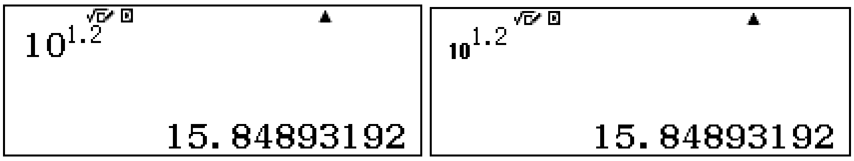
Adjusting and scrolling the table suggests that the answer to the first question is about 2021, which is 7 years after 2014. The answer to the second question seems to be around 14.6 or 14.7 hours after the start. While these kinds of approaches may be practically adequate, there is a more powerful approach, using logarithms, which we will now consider.

Logarithms

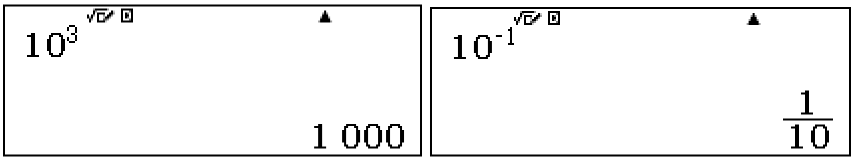
The idea of a *logarithm* is strongly related to the idea of an exponent. To see how they are related, consider the exponential function with 10 as the base:

$$f(x) = 10^x$$

If we know the exponent, such as $x = 1.2$, we can find the value using the calculator with the [$x^{\blacksquare}$] key. In fact, there is a special command for exponential functions with a base of 10, $(10^{\blacksquare})$, obtained with [SHIFT] [$\log_{\blacksquare}\square$]. The screens below show both methods.

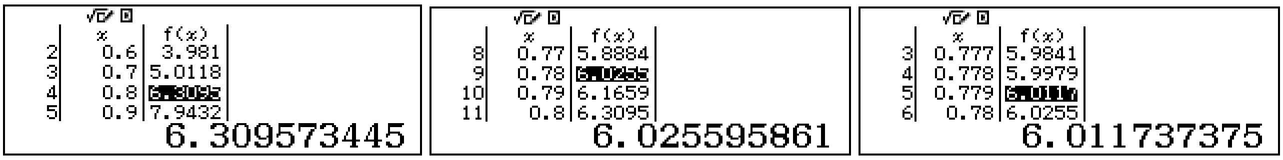


For some familiar integer values of x, it is not even necessary to use the calculator to find the function value, although these are shown below.

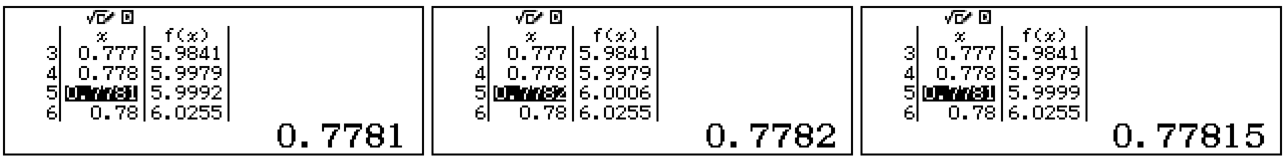


For numbers that are powers of ten it is relatively easy to see the relationship between the exponent and the value, but it is much more difficult to find the exponent of 10 that will give a particular value for other numbers, however. For example, what exponent is needed for $f(x) = 10^x = 6$?

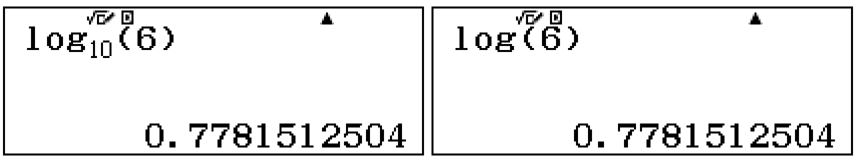
One way to address this problem is to use a table of values for the function. Look carefully at the following screens to see that each one gets closer to finding the value for which $10^x = 6$.



The value seems to be between $x = 0.778$ and $x = 0.779$. Further tables can be constructed to improve the accuracy of the value. An alternative approach to masking new tables is to keep changing the x-value in a table, tapping [=] to get the function value, and adjusting the x-value until $f(x)$ is close to 6. There are some successive examples below.

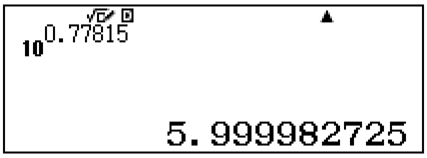


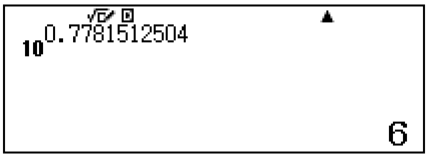
Finding the exponent to give a particular value is such a helpful tool, however, that the calculator has special keys to complete it efficiently. The value is referred to as the *logarithm* of 6 to the base of 10. The screens below show that $\log_{10}6 = 0.77815$, correct to five decimal places. In the first screen, the [$\log_{\blacksquare}\square$] key has been used. You need to enter the base of the logarithms (10, in this case).

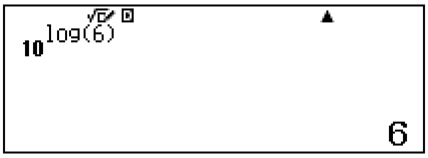


Logarithms to base 10 are so widely used that they are often described simply as logs (without referring to the base), and the calculator uses the same convention. The second screen above shows this, using the command SHIFT (−) (log). With this command, it is not necessary to enter base 10.

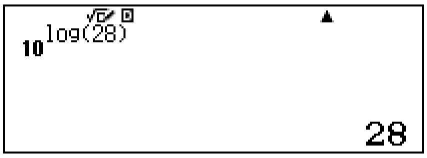
The logarithm of a number is the exponent to which 10 must be raised to produce the number. The first screen below shows that the value of 0.77815 obtained in the tables is a good approximation to log 6, the middle screen shows that the value of 0.7781512504 obtained from the log key is a better approximation, while the third screen shows the exact meaning of log 6. An approximation is usually used in practice, as most logarithms are irrational numbers, as in this case.

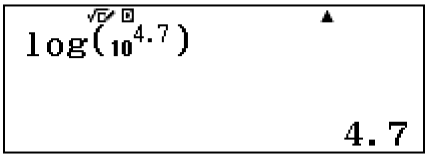






Check with your calculator that the two inverse operations of raising 10 to an exponent and finding the exponent to which 10 is raised to get a particular number are closely related, showing the strong connection between exponential and logarithmic functions. To illustrate, the two screens below demonstrate the close relationship between these two functions. Study these screens carefully, as each reflects the definition of a logarithm to base 10.





Logarithms to base 10 are also described as 'common' logarithms and were very important for hundreds of years for completing calculations – before the invention of calculators and computers. Their properties were used to construct and use slide rules, also used for hundreds of years.

Properties of logarithms

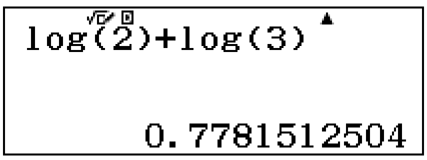
It is no longer necessary to use logarithms for computation, especially when you have a calculator. However a brief examination of how logarithms were once used for computation will make some properties of logarithms clear.

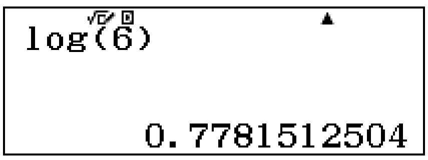
The key properties of logarithms that made logarithms so useful in the past follows from the index laws, illustrated below:

$$10^{3.1} \times 10^{1.2} = 10^{3.1+1.2} = 10^{4.3} \text{and} \left(10^{1.2}\right)^3 = 10^{1.2\times3} = 10^{3.6}$$

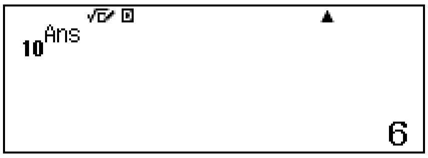
These statements illustrate that, when numbers are written as powers of 10, they can be multiplied by adding the powers. Similarly, a power of a number can be found by multiplication. It is much easier to add than to multiply, and much easier to multiply than find powers, so that before the calculator age, representing numbers using logarithms resulted in much easier calculations for everyone, including especially scientists and engineers. (Logarithms were obtained from printed tables, not a calculator, of course. Slide rules were also used to get approximate answers.)

To give an easy illustration of this process, consider the two screens below.



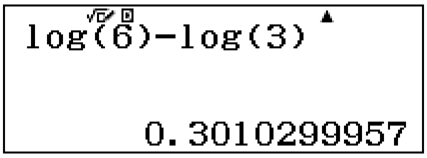


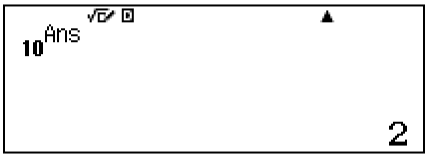
Notice that log 2 + log 3 = log 6, since each is the same number (≈ 0.778). Although 3 x 2 =6 involves a multiplication, only addition is needed when using the logarithms. To get the result of 6, it is necessary to be able to convert the result from a logarithm (in this case 0.778 …) back to a number, which is completed on the calculator by using the result as an exponent, as shown below:



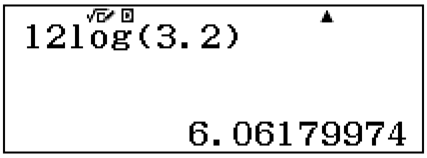
Previously, this reverse process was also carried out using a book of tables, not a calculator.

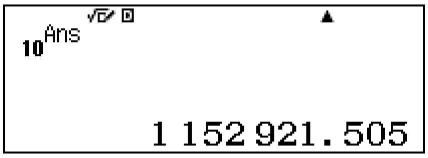
A similar idea applies to division, handled by subtracting logarithms, to show that 6 ÷ 3 = 2.

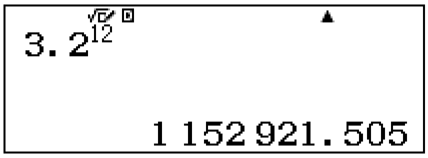




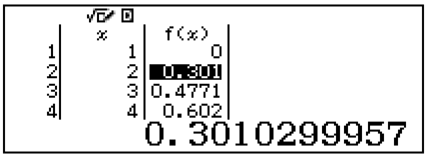
Finding powers and roots of numbers involved multiplication and division, and was much easier with logarithms than with other methods. In the screens below, for example, the twelfth power of a number is found by multiplying its logarithm by 12. This is a much easier way of calculating 3.2^{12} than multiplication of 3.2 by itself twelve times. Of course, these days, it is even easier to use the calculator directly, as the third screen below shows.

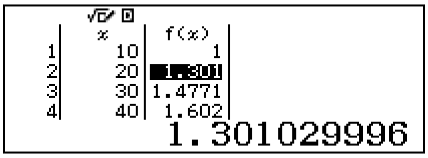


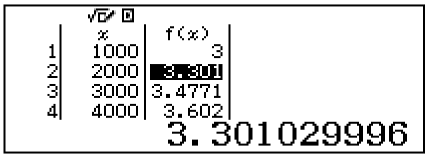




Logarithms to base 10 were especially useful, as we use a decimal number system (which also has a base of 10). The tables below show some examples of the logarithmic function $f(x) = \log x$.







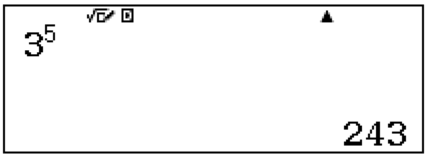
Notice that log 20 = log 10 + log 2 = 1 + log 2 and log 2000 = log 1000 + log 2 = 3 + log 2. For this reason, logs of large numbers could be obtained easily from logs of smaller numbers, by adding on a term associated with a power of 10, and so tables of logarithms frequently listed only the numbers between 1 and 10.

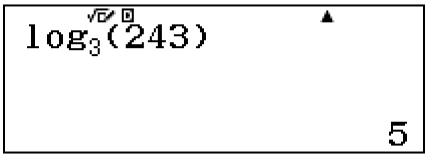
As noted, your calculator now handles the computations for which previously logs were necessary. Find an old table of logarithms and check for yourself some of the properties described here.

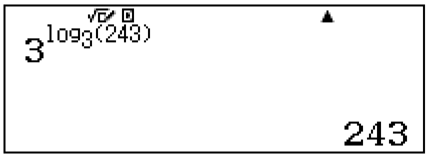
Logarithms to other bases

The idea of a logarithm is connected to an exponential function. To date, we have used only exponential functions with a base of 10. But any positive number can serve as a base. To illustrate, since $3^5 = 243$, we could say that the *logarithm to base 3* of 243 is 5, written as $\log_3 243 = 5$.

On the calculator, logs to any positive bases can be obtained with the [log▪□] key, as you have already seen for base 10. Enter the base of the logarithm and then tap ▶ to enter the number concerned, before tapping = to get the result:

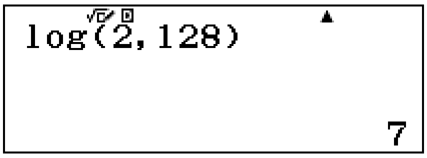


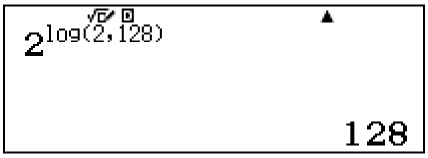




The final screen above shows the relationship between the logarithm to a particular base and the exponential function with the same base is similar to that for base 10.

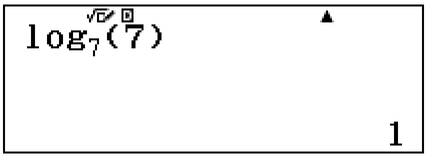
Logs to other bases are also available via the other logarithm key, using SHIFT (−). So far, we have used this command to obtain logs to base 10, but logs to other bases are available by typing the base, followed by a comma (obtained with SHIFT)) as shown below to obtain $\log_2 128 = 7$:

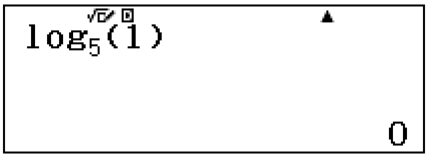


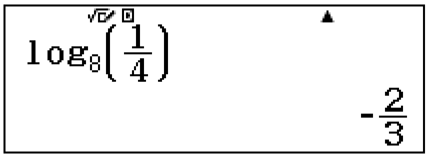


(When no base is entered, it is assumed to be 10.) Either method produces the same result, although the use of the $\log_\square\square$ key is usually a little quicker to enter. The choice of method is a matter of personal preference.

Notice also, for any positive base, the log of the base itself is always 1, the logarithm of 1 is always 0 (since any base raised to the power of 0 is 1) and the logarithm of a positive number less than 1 is negative. Here are three examples of these properties.







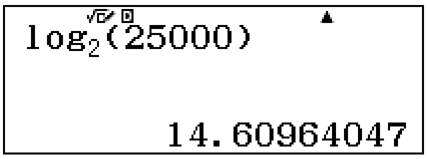
We suggest that you try some more examples for yourself to check these properties. Check also that you cannot find logarithms of negative numbers or logarithms to a negative base, as neither of these is mathematically defined.

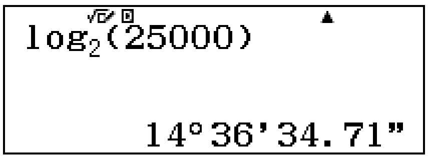
Logarithms provide a powerful tool to solve some practical problems, such as those suggested in the case of the bacteria and the population of Saudi Arabia above. To find when the bacteria population is expected to reach a million involves solving the exponential equation:

$$40 \times 2^x = 1\ 000\ 000$$

$$\text{or } 2^x = 25\ 000$$

You can think of this problem using logarithms to base 2, as the solution (x) is the logarithm to base 2 of 25 000. The $\log_\square\square$ command on the calculator allows this to be determined readily:





The solution is about 14 hours and 36 minutes. This is a much easier method of solution than using the table shown earlier. (Notice that the result of 14.61 hours has been converted above to hours, minutes and seconds, using the °'" key that you saw in the Module 4, although excessive accuracy is not warranted here by the model used.)

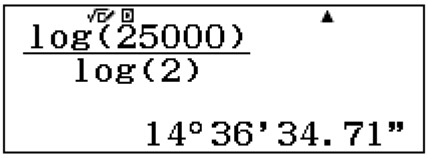
There is another way to solve equations like $2^x = 25\ 000$, involving finding the logarithms of each side of the equation (using the property of logarithms that $\log a^b = b \log a$)

$$\log (2^x) = \log 25\ 000$$

$$x \log 2 = \log 25\ 000$$

$$\text{So } x = \frac{\log 25000}{\log 2}$$

This result is the same as shown earlier, although it is a little more cumbersome to obtain:

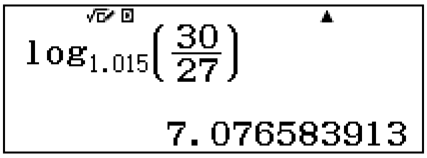


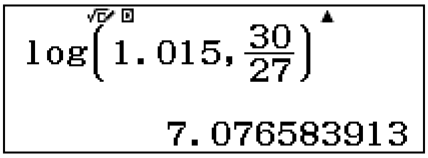
To find when the population of Saudi Arabia is predicted to be 30 million also involves solving an exponential equation:

$$f(x) = 27 \times 1.015^x = 30$$

$$\text{That is, } 1.015^x = \frac{30}{27}$$

You can think about this problem using logarithms, as the equations shows that the value sought can be regarded as the logarithm of 30/27 to the base 1.015. The calculator allows you to find this value directly, as shown below, using either of the two logarithm commands.





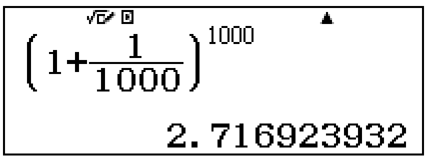
So the population of Saudi Arabia is predicted to reach 30 million a little over 7 years after 2014, or about 2021. Again, this is an easier solution process than using the table of values of the exponential function shown earlier.

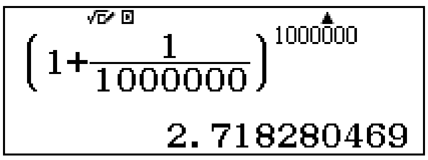
e and natural logarithms

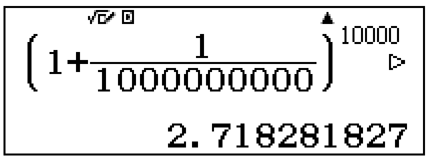
Many exponential functions involve the mathematical constant *e*. This important irrational number can be defined in many ways, including as the limiting value, as *n* becomes infinitely large, of

$$\left(1+\frac{1}{n}\right)^n.$$

The next three screens show that, as *n* gets larger, the value seems to slowly get closer and closer to the accepted value of $e \approx 2.718281828459\ \ldots$



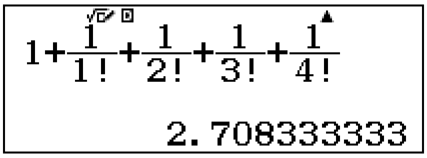


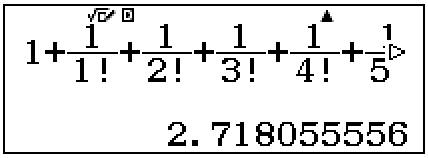


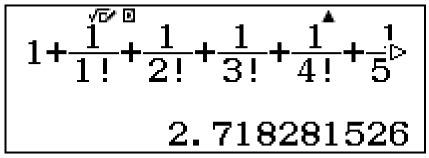
Another way of defining *e* is via an infinite series:

$$e \approx 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+...$$

You can approximate *e* with the first few terms of this series on your calculator, as shown below (The factorial command [x!] is obtained with [SHIFT] [x^{-1}]):

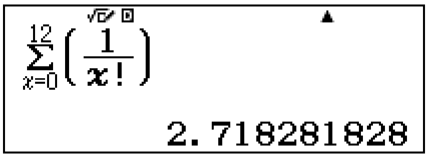




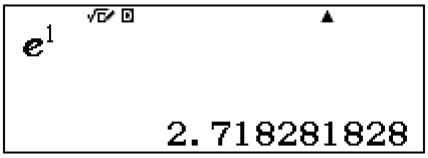


Although they might look the same, these screens evaluate the first 4, 6 and 9 terms respectively of the series. They show that the series gets close to the accepted value for *e* quite quickly (i.e., with not many terms needed).

In Module 14, you will see a more efficient way to evaluate series like this with a single command, as shown below. This is much easier than entering successive terms. With twelve or more terms, the series gives the same result as shown below, correct within the limitations of the calculator screens.



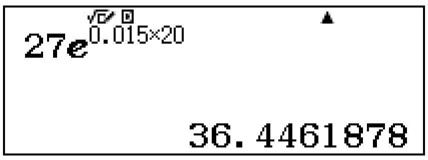
The constant *e* is so important that values of the exponential function $f(x) = e^x$ are available directly on the calculator with the ($e^{\blacksquare}$) key, obtained with [SHIFT] [ln]. In fact, the calculator uses a series like the one above to evaluate this function. When $x = 1$, a value for *e* itself is given:



The number *e* is used widely in mathematics. A good example involves constructing models for continuous change, which is approximated by natural growth and decay processes. If an initial quantity *P* grows or decays continuously at an annual rate of *r*, then the amount $f(t)$ after *t* years is given by

$$f(t) = Pe^{rt}$$

Thus, in the example of Saudi Arabia's population earlier, the annual rate *r* is 1.5% = 0.015. So a continuous growth model requires $P = 27$, $r = 0.015$. According to this model, the screen below shows that the population after 20 years will be about 36.4 million, which is close to the value given earlier.



The results are not identical, because the Saudi Arabian population growth is not continuous, although it is a close approximation to it.

Logarithms to the base of *e* are also important in many applications of mathematics, so that these are available directly on the calculator with the [ln] key. The abbreviation ‘ln’ stands for *natural logarithm*, as logarithms to base *e* are usually called.

Natural logarithms have many of the same properties as other logarithms, noted earlier. Here are some examples:

ln(1)	ln(2)+ln(3)	ln(6)
0	1.791759469	1.791759469

The following two screens show that the exponential function and natural logarithmic function are inverses of each other (similar to the situation for 10 with common logarithms):

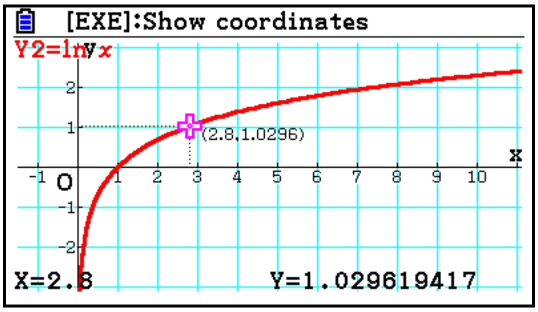
$\ln(e^{4.5})$	$e^{\ln(7.3)}$
4.5	7.3

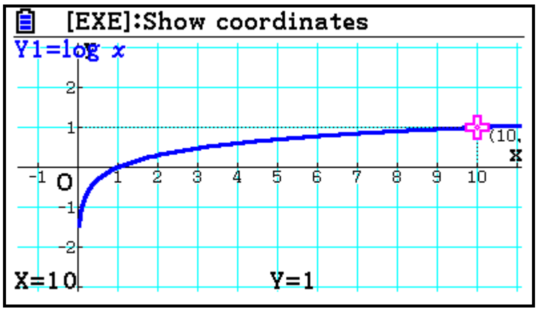
Similarly, the relationships of logarithms to other bases noted earlier to find $\log_2 25000$ hold for both common and natural logarithms:

$\log_2(25000)$	$\frac{\log(25000)}{\log(2)}$	$\frac{\ln(25000)}{\ln(2)}$
14.60964047	14.60964047	14.60964047

Notice that each of the three procedures shown here gives the same result.

Finally, graphs of logarithmic functions look similar for both common and natural logarithms, as shown below, using graphs from the CASIO fx-CG20 graphics calculator.





In each case, both graphs go through (1,0), both have negative values for $x < 1$ and both are undefined for $x < 0$. Both graphs have a positive slope that is decreasing as x increases. Each graph is a reflection about $y = x$ of its corresponding inverse function, $f(x) = e^x$ and $f(x) = 10^x$ respectively.

Use your calculator to explore and compare these properties though tables of values. For example, study the tables below which show some values for $f(x) = \ln x$ and $g(x) = \log x$ consistent with graphs above. Note that logarithms are only defined for positive numbers and ln 1 = log 1 =0.

	x	f(x)	g(x)
1	0	ERROR	ERROR
2	0.5	-0.693	-0.301
3	1	0	0
4	1.5	0.4054	0.176

-0.3010299957

	x	f(x)	g(x)
1	2.7	0.9932	0.4313
2	2.8	1.0296	0.4471
3	2.9	1.0647	0.4623
4	3	1.0986	0.4771

1.029619417

	x	f(x)	g(x)
2	8	2.0794	0.903
3	9	2.1972	0.9542
4	10	2.3025	1
5	11	2.3978	1.0413

1

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. Evaluate $\sqrt[5]{27}$.

2. Evaluate $13^{\frac{7}{9}}$.

3. (a). Write $16^{\frac{3}{4}}$ in two separate ways, each using radical signs (√). Check that these lead to the same numerical result.

   (b). Evaluate $16^{\frac{3}{4}}$ directly on the calculator and check that the result is the same as in part (a).

4. A population of cells doubles in size every day. After x days, the population is given by $P(x) = 60 \times 2^x$.
   (a) What is the size of the initial population?
   (b) How many cells will there be after 12 days?
   (c) After how many days will there be two million cells?

5. Which of the following two exponential models increases at a faster rate: $f(x) = 3 \times 4^x$ or $g(x) = 12 \times 3^x$?

6. A Chinese city has a population of 2.1 million and is growing in population at the rate of 1.8% per year.
   (a) Write the annual population in the form of an exponential model.
   (b) Use a table of values or the CALC command to find the expected population of the city in 10, 20, 30 and 40 years from now.

7. (a) Find $\log_{10}81$ and $\log_{10}9$.
   (b) Explain why $\log_{10}9 < 1$.
   (c) Explain why $\log_{10}81$ is twice $\log_{10}9$.

8. Evaluate:
   (a) $\log 10^{0.6}$
   (b) $10^{\log 2\pi}$

9. (a) Evaluate $\log\frac{9}{5}$

   (b) Write $\log\frac{9}{5}$ as a difference of two logs, and use this to check your answer to part (i).

10. (a) Evaluate $\log_4 7$ and $\log_4 28$
    (b) Explain the relationship between the two values in part (a).

11. Evaluate $\log_{16}1$.

12. Solve $3^x = 143$, using the $\log_{\blacksquare}\square$ key.

13. (a) Find log 5 and ln 5.
    (b) Which of these two is larger? Explain why it should be expected to be larger.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. (a) Use tables of values to compare the exponential models $f(x) = 0.8^x$ and $g(x) = 1.25^{-x}$.
   (b) Explain the relationship between these two models.
   (c) Find another pair of exponential models that are related in the same way as these are.

2. The function $m = 1000 \times 1.029^{-t}$ can be used to model the radioactive decay of one kilogram of plutonium. In the model, m is the number of grams of plutonium remaining and t is measured in *thousands* of years.
   (a) How much of one kilogram of plutonium is left after 10 thousand years?
   (b) Make a table of values to see how much plutonium would remains after every 5000 years up to 40 000 years. What is the approximate half-life of the plutonium? (That is, after how many years will it have decayed to half its original mass?)
   (c) A typical 1000 Megawatt nuclear reactor produces about 230 kg of plutonium per year. How much of this plutonium would remain if it were left in a cooling pond for 40 years?

3. The world population P (in billions) in the twentieth century has been estimated as below:

Year t	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Population P	1.65	1.75	1.86	2.07	2.30	2.56	3.04	3.71	4.46	5.28	6.08

   It has been suggested that the population P (billions) can be modelled with the exponential function: $P = 8.7 \times 10^{-12} \times e^{0.0136t}$ in the year t.
   (a) Use CALC or a table of values from $t = 1900$ to determine how well this model matches the data.
   (b) What will the model predict the population to be in this year?
   (c) When does the model predict the world population to be eight billion?

4. (a) How do the common logarithms of numbers compare with the logarithms of squares of the numbers? Try some examples to look for a relationship (such as log 7 and log 49, log 13 and log 169, log 10 and log 100)?
   (b) Compare the logarithms of numbers with the logarithms of their cubes.
   (c) What is the effect of taking logarithms to a different base in parts (a) and (b)? E.g., how do logarithms to base 4 of numbers compare with logarithms to base 4 of their squares?

5. A microbiologist was studying the growth of a virus. Consider the following experimental data for the number (y) of cells she observed in a culture after x days:

x days	1	2	3	4	5
y cells	15	46	134	400	1220

   (a) Use graph paper to plot y against x. What is the shape of the graph?
   (b) Use graph paper to plot $\log y$ against x. How has the shape of the graph changed?
   (c) Use graph paper to plot $\log_3 y$ against x. Describe the shape of the graph.
   (d) Use your graphs to suggest a relationship between x and y.

6. Construct some tables of values to compare logarithmic functions like $f(x) = \log_a x$ for various bases (a) and for $0 < x < 10$. Use at least three different bases. Compare some tables to answer these questions:
   (a) For which value of x do all the functions have the same value?
   (b) For which values of x are the values of $f(x)$ negative?
   (c) For which values of a do the values of $f(x)$ increase most rapidly?
   You may find it helpful to use your tables of values to sketch some graphs quickly.

Notes for teachers

This module highlights the ways in which the CASIO *ClassWiz* can support students to understand exponential and logarithmic functions and use them in applications of mathematics. The module makes use of Table mode and various logarithmic and exponential keys. The text of the module is intended to be read by students and will help them to see how the calculator can be used in various ways. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. 1.933 2. 7.352 3. (a) $\sqrt[4]{16^3} = \left(\sqrt[4]{16}\right)^3 = 8$ 4. (a) 60 (b) 245 760 (c) about 15 5. $f(x)$
6. (a) $P(x) = 2.1 \times 1.018^x$ (b) 2.51, 3.00, 3.59, 4.29 (millions) 7. (a) 1.908, 0.954 (b) $\log_{10}9 < \log_{10}10$ and $\log_{10}10 = 1$ (c) $\log_{10}81 = \log_{10}9^2 = 2\log_{10}9$ 8. (a) 0.6 (b) 2π 9. (a) 0.255 (b) $\log 9 - \log 5$ 10. (a) 1.404, 2.404 (b) $\log_4(7\times4) = \log_4 7 + \log_4 4 = \log_4 7 + 1$ 11. 0 12. $\log_3 143 \approx 4.517$
13. (a) 0.699, 1.609 (b) ln 5 is larger because the base of the logarithms is smaller.

Activities

1. The activity draws attention to the two different ways of representing exponential decay: with a negative index (and a base larger than 1) or a positive index with a base less than 1. Students comfortable with laws of indices should be able to explain why these two alternatives give the same values in this case and in general. [Answers: (a) the tables are the same; (b) the models are the same; (c) other pairs can be found by choosing bases that are reciprocals of each other.]

2. Radioactive decay is an important application of exponential functions. Students can answer these questions by exploring a table, although more sophisticated students may answer part (b) using $\log_{1.029}0.5$. Care is needed to remember that t is measured in thousands of years. A graph may help students to visualise decay. [Answers: (a) 751.4 g (b) about 24 200 years (c) 229.7 kg]

3. Population models are important examples of exponential functions. This example uses e to demonstrate its wide utility. Discuss with students the assumptions underlying models of this kind (especially the (false) assumption that growth patterns stay the same over time). The model does not fit the data precisely, but nonetheless provides an important sense of changes. Similar models might be explored for the students' own country if data can be accessed online. [Answers: (b) model predicts 6.93 billion for 2015, which is a bit low (c) late in 2025.]

4. This activity invites students to explore the power relationship that, in general, $\log x^k = k \log x$. This relationship holds regardless of the base for the logarithms. [Answers: (a) logs of squares are twice logs of original numbers (b) logs of cubes are 3 x logs of numbers (c) same for all bases.]

5. This activity requires students to plot points on a graph. Care is needed to choose suitable scales, especially for part (a). Students should see the linearising effects of log transformations, which underpin exponential modelling in statistics and are the basis for semilog graph paper. Most students will need help with part (d), unless they are familiar with using logarithms in this way. [Answers: (a) exponential (b) linear (c) linear with slope = 1 (d) relationship is close to $y = 5 \times 3^x$ This is perhaps easiest to see from graph (b) which seems to show $\log_3 y \approx 1.46 + x$]

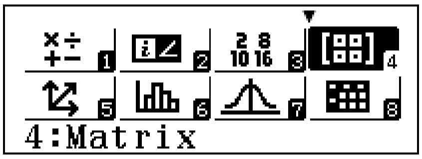
6. It would be helpful for students to draw some graphs here, as these are a little easier to interpret than the tables and a record can be made for comparisons. [Answers: (a) for $x = 1$, $f(x) = 0$
(b) $0 < x < 1$ (c) the smaller the value of a, the steeper the increase.]

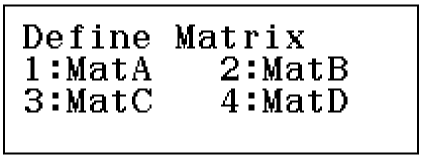
Module 7
Matrices

Matrices are very useful in mathematics to represent information such as a system of equations or a set of points, and to manipulate the information efficiently. This module requires the use of Matrix mode, accessed by tapping [MENU] [4] on the CASIO *ClassWiz*.

Defining matrices

A matrix is a rectangular array of numbers, organised into rows and columns. The calculator allows you to define up to four separate matrices, named A, B, C and D. Each matrix can have 1, 2, 3 or 4rows and 1, 2, 3 or 4 columns. When Matrix mode is entered, you must choose which matrix you wish to define, as shown below.

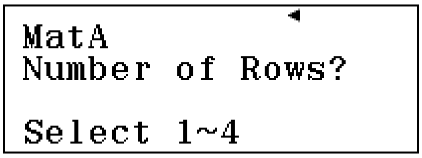


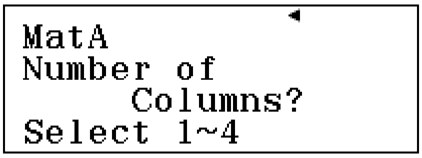


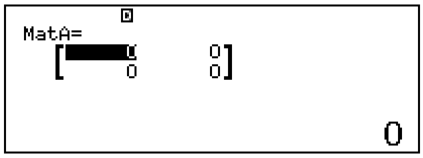
Tap [1] to define Matrix A, which is abbreviated on the calculator to *MatA* and in this module, using bold type, to **A**. To define a matrix, its dimensions (or size) needs to be chosen, following the calculator screen requests for the numbers of rows and columns. By convention, rows are mentioned first, so a 3 x 2 matrix has three rows and two columns. This (square) matrix has dimensions 2 x 2:

$$\mathbf{A} = \begin{bmatrix} 3 & \frac{1}{2} \\ -2 & 4 \end{bmatrix}$$

To enter this matrix into the calculator, first define **A** as a 2 x 2 matrix:

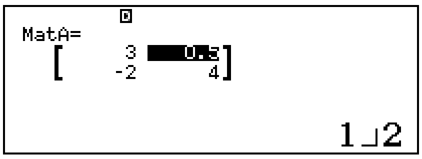






Enter the matrix coefficients into the empty matrix, tapping [=] after each one. You can use the cursor keys to move to a different cell in the matrix to correct errors or change terms.

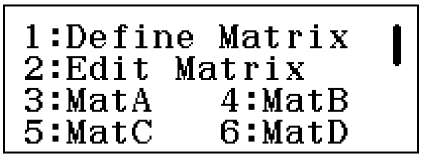
If you want to enter a fraction in a cell, tap the numerator, then the $\frac{\square}{\square}$ key and then the denominator. For example, the screen below shows the second element in the first row, $\mathbf{A}_{12} = ½$.



Notice the value for a highlighted matrix element is shown at the bottom of the screen.

When you have finished defining a matrix, tap [AC].

To define other matrices, or to change the values in a matrix, select the [OPTN] menu.

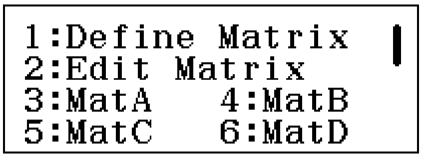


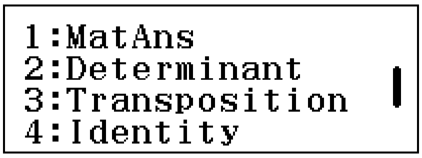
To change the dimensions of a matrix, you will need to define it again. Tap [AC] when finished.

Note that if you leave Matrix mode, any matrices already defined will be erased.

Matrix arithmetic

Once matrices have been defined, they can be retrieved and various kinds of matrix arithmetic can be carried out on the calculator. These all use the [OPTN] menu. (Tap ▲ and ▼ to move between menu pages.)

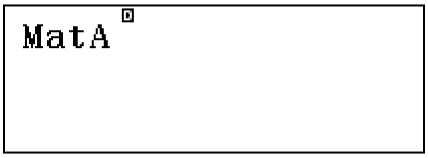


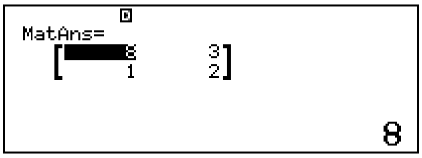


In this module, we use bold type to show a matrix. Thus Matrix A will be represented as **A**. To illustrate matrix arithmetic, define the following four matrices in your calculator:

$$\mathbf{A} = \begin{bmatrix} 8 & 3 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 2 \\ -1 & 7 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 8 & 5 \\ 12 & 9 \\ 3 & 4 \\ 6 & 8 \end{bmatrix}$$

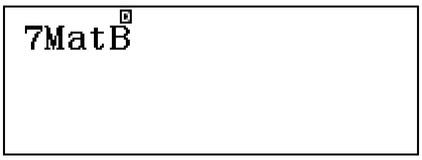
To display a matrix (**A**, **B, C** or **D**), select it from the [OPTN] menu and tap [=]. (The first screen below shows the calculator before [=] is tapped, while the second shows the result after [=] is tapped.)

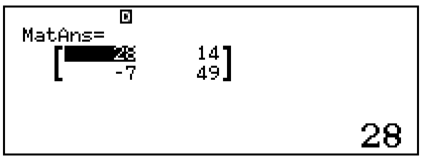




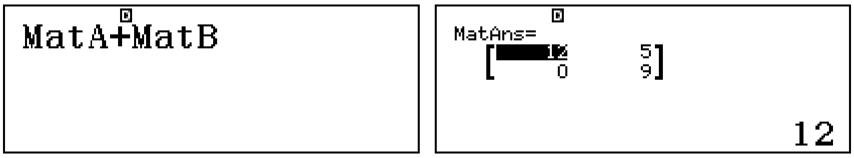
Notice that, as for the calculator itself, the most recent result of a matrix command is shown as *Ans*. This result can be retrieved using *MatAns* by tapping [1] in the second page of the [OPTN] menu.

A matrix can be scaled by multiplying it by a number, which has the effect of multiplying each matrix element by the number (which is called a *scalar* number because it is not itself a matrix). It is acceptable, but unnecessary, to enter a multiplication sign for a scalar multiple. The screens below show the result of scaling **B** by 7 to obtain the new matrix, 7**B**:





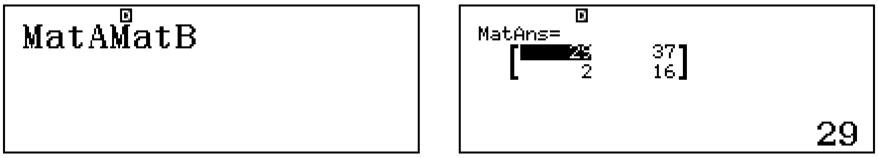
Two matrices with the same dimensions, such as **A** and **B**, can be added or subtracted. This results in a new matrix in which each of the elements in the same position in the original matrices have been calculated in the same way. For example, notice below that 8 + 4 = 12 and 2 + 7 = 9:



Matrices that do not have the same dimensions cannot be added or subtracted, however, and will result in a dimension error, because some elements of one will not match elements of the other. Check this for yourself by using the calculator to find **A** + **C**.

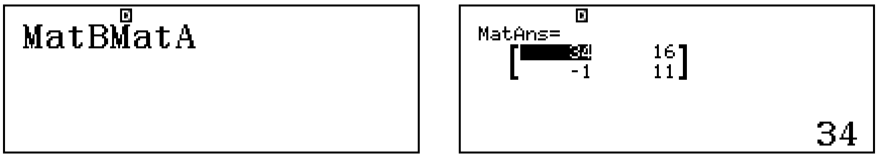
Similarly, multiplication of two matrices is only defined when the number of columns in the first matrix is the same as the number of rows in the second matrix. The first element in the product **AB** comes from multiplying the first row of **A** by the first column of **B**, and then adding the results:

$$8 \times 4 + 3 \times -1 = 29$$

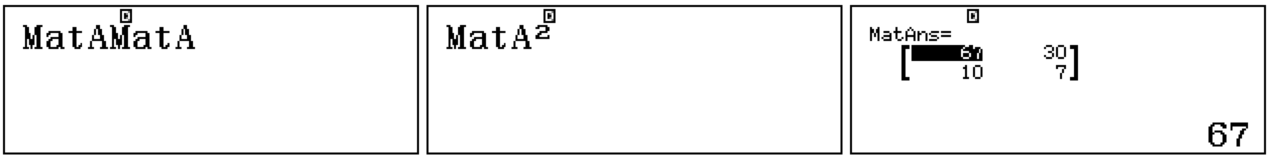


Similarly, $\mathbf{AB}_{12}$ is obtained by multiplying the first row of **A** by the second column of **B**. Check the other entries for yourself by hand.

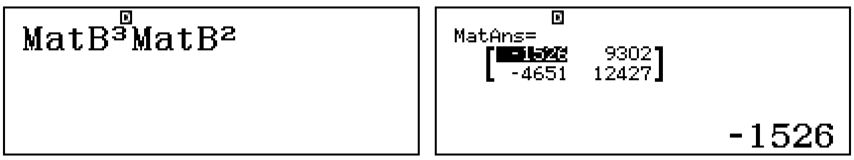
Matrix multiplication can be written using a multiplication sign, although this is not necessary. Notice from the result below that matrix multiplication is not *commutative*; that is **AB** does not give the same result as **BA**.



A square matrix can be multiplied by itself (since it has the same number of rows and columns). You can use the x^2 and x^3 (i.e. SHIFT x^2) commands to find the square and cube respectively of a square matrix. Two ways of finding $\mathbf{A}^2$ are shown below, with the second using the x^2 key.



Notice that you cannot use the power key $x^{\blacksquare}$ for this purpose however (since this does not involve the calculator multiplying something by itself, but uses different mathematical operations using logarithms). If you want to obtain higher integer powers of a matrix than the third power, you can do so with combinations of squaring and cubing.



To illustrate this, the screen above shows how to find $\mathbf{B}^5$, the fifth power of **B**.

Matrix inversion

Unlike numbers, matrix division is *not* defined directly. Instead, *inverses* of matrices are used. This is only possible for square matrices – those with the same number of rows and columns.

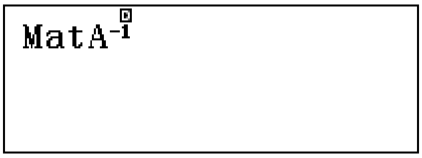
You can think about this as similar to division for numbers. For example, $7 \div 8 = 7 \times 8^{-1}$. The (multiplicative) inverse of 8, represented by 8^{-1}, is the number that you need to multiply 8 by to obtain a result of 1, which is the *identity* for multiplication:

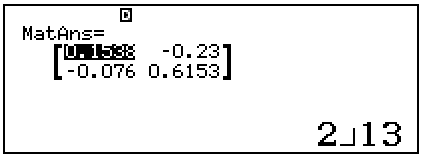
$$8 \times 8^{-1} = 1.$$

In a similar way, the inverse of matrix **A**, represented by $\mathbf{A}^{-1}$ has a similar property using the identity for matrices, which consists of 1 in each of the diagonal elements and zero elsewhere:

$$\mathbf{AA}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On the calculator, an inverse of a square matrix can be obtained using the same inverse key x^{-1} as for numbers, as shown below:

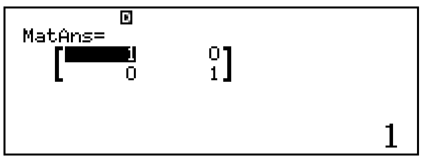




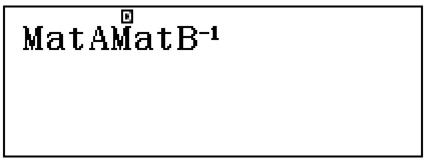
If you move the cursor to the various elements of $\mathbf{A}^{-1}$, you will see them shown as fractions. In this particular case, check for yourself that

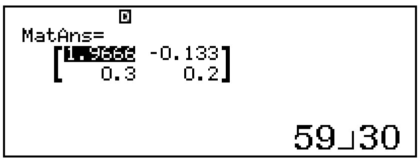
$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{13} & -\frac{3}{13} \\ -\frac{1}{13} & \frac{8}{13} \end{bmatrix}$$

Also check for yourself how the inverses work by using the calculator to see that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$**,** where **I** stands for the identity matrix.

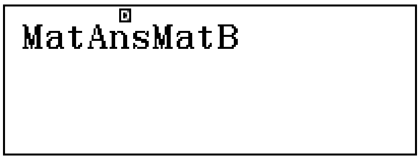


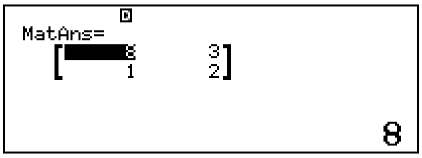
Hence, to obtain **A**÷**B**, we need to use the calculator to determine $\mathbf{AB}^{-1}$, as shown below:



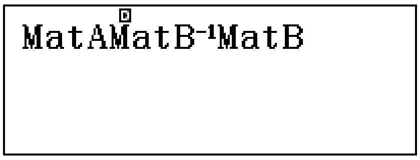


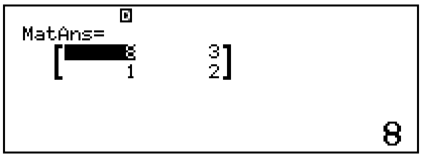
In the same way that multiplying 7÷8 by 8 produces the original number (7), multiplying $\mathbf{AB}^{-1}$ by **B** produces a result of **A** again. You can see this by multiplying the above result immediately by **B**, as shown below. Notice that the calculator uses *MatAns* in that case:





You could also see that $\mathbf{AB}^{-1}\mathbf{B} = \mathbf{AI} = \mathbf{A}$ by entering the entire calculation on a fresh screen:



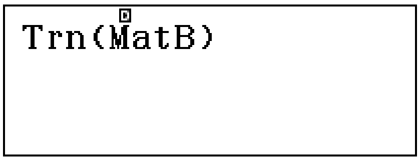


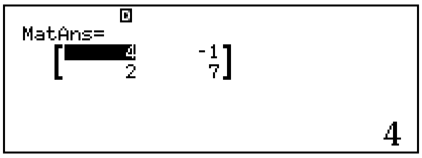
Notice that, since matrix multiplication is not commutative, it is necessary to multiply matrices in a particular order. In this case, check for yourself that $\mathbf{BAB}^{-1}$ does *not* result in **A**.

The calculator uses a number called the *determinant* of a (square) matrix in order to calculate its inverse. The determinant of a matrix is related to the sizes of the elements, and can be calculated directly from the second page of the OPTN menu. In the case of **A**, if you study the elements of the inverse and the original matrix, you will be able to see how the determinant of 13 is involved, although it is harder to see the connection for 3 x 3 and higher matrices. (See Activity 5).

In the same way that the number 0 does not have a multiplicative inverse, matrices with a zero determinant do not have an inverse.

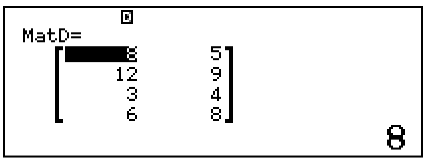
The *transposition* (or *transpose*) of a matrix is a new matrix with the rows and columns reversed. The transpose of **B** is usually represented as **B'**. This can be obtained directly from the second page of the OPTN menu, as shown below.



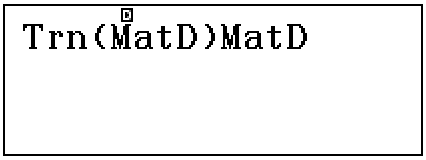


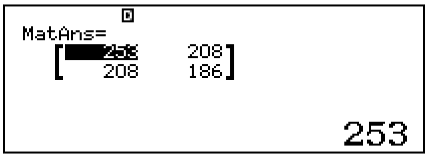
Transposes are used in statistics because of a useful result that pre-multiplying the transpose of a matrix by the matrix itself gives a symmetric matrix with the sums of squares in the diagonal and the sums of cross products elsewhere. Computer packages use this property for fast and efficient data analysis.

Consider the 4 x 2 matrix **D** (defined earlier and shown below). Suppose that each of the two columns shows the scores of four people on each of two variables. So each column represents a variable, while each row represents the scores of a single person on the two variables.



You can check the property for yourself by using the calculator to obtain **D'D**, as below.





The sums of the squares of the columns of **D** are $8^2 + 12^2 + 3^2 + 6^2 = 253$ and $5^2 + 9^2 + 4^2 + 8^2 = 186$, which are the two diagonal terms of **D'D**. The sum of cross-products (between columns) in this simple case is defined as $8 \times 5 + 12 \times 9 + 3 \times 4 + 6 \times 8 = 208$. Notice that this is the off-diagonal term of **D'D**, symmetric on each side of the matrix.

In the case of larger matrices, the same idea holds. If each column of the matrix represents a variable, the diagonal terms are used by computers to find variances for each variable and the off-diagonal terms are used to find correlation coefficients between variables, as described in Module 11, concerned with bivariate data analysis.

Transformation matrices

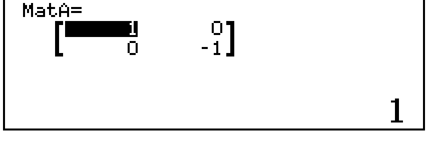
Matrices are especially useful to describe transformations in the plane and in space. For example, consider the special 2 x 2 matrix below used to *premultiply* to find the images of three points (-4,3), (5,1) and (3,-2), noticing that the points are represented with 2 x 1 column vectors.

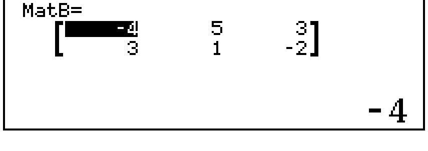
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix}=\begin{bmatrix} -4 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}=\begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Study these carefully to see the pattern: in each case, the x-value of the image is unchanged, while the y-value is reversed in sign. So, the matrix is special because it has the effect of reflecting the points about the x-axis. Other 2 x 2 matrices will also result in transforming points, but with different geometric effects of course.

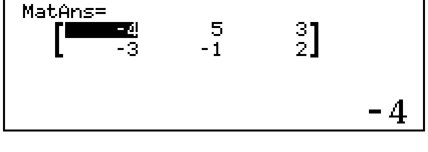
In this illustrative case, the numbers are easy enough to do the matrix multiplication in your head, but other situations involving more complicated transformation matrices are better done with the calculator. One way to do this is to define the transformation matrix as a 2 x 2 matrix and each of the three points as a separate 2 x 1 matrix. However, this is a little tedious and also problematic as the calculator will allow you to have only three matrices defined at any one time.

An easier and efficient way is to define the 2 x 2 transformation matrix and then define a 2 x 3 matrix that represents all three of the points simultaneously, with one point in each column of the matrix. The screens below show the transformation matrix as **A** and the triangle of three points represented by **B**. (Now we are using different matrices **A** and **B** from previously).





The result of the transformation is readily seen by calculating **AB**:



The columns of the resulting matrix show the transformed points in columns, (-4,-3), (5,1) and (3,2), showing that the triangle has been reflected about the x-axis.

Matrices and equations

An important use of matrices is to represent and then to solve systems of linear equations. If you have studied Module 4, you will have seen that the calculator uses a matrix of coefficients to solve a system of linear equations. Precisely the same task can be performed using matrices.

To illustrate, consider the following system of linear equations:

$$3x - y = 20$$
$$2x + 5y = 2$$

This can be represented as a matrix equation **AX** = **B** with

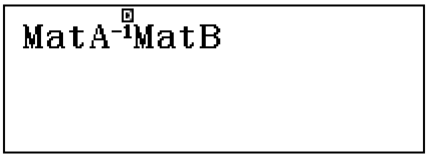
$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 20 \\ 2 \end{bmatrix}.$$

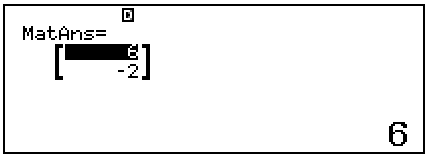
Then the matrix equation can be solved by pre-multiplying both sides of the equation by the inverse of the matrix of coefficients, $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\text{i.e., } \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

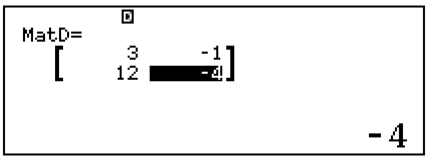
After entering Matrices **A** and **B** into the calculator, the solution is obtained readily with $\mathbf{A}^{-1}\mathbf{B}$ as shown below:





The solution of x = 6 and y = -2 can be read directly from the 2 x 1 matrix above. Check by substitution mentally that this solution fits each of the two original equations.

Notice that coefficient matrices that do not have an inverse (or with determinant of zero) are associated with systems of equations that do not have a (unique) solution. Here is an example:



Can you see why there is no solution in this case?

Similar processes are involved for using matrices to find the solution to a system of three linear equations in three unknowns or four linear equations in four unknowns, and so on.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. Given the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 4 & 5 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

calculate where possible on the calculator (and if it is not possible, explain why not):
(a) 5**A** (b) **A** + **C** (c) **AB** (d) **CD** (e) **BD** (f) $\mathbf{A}^2$ (g) $\mathbf{A}^{-1}$ (h) $\mathbf{C}^6$

2. (a) Use the calculator to find the determinant of $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & \frac{1}{2} \end{bmatrix}$

(b) Edit matrix **A** to replace the fraction with 3 and find the determinant of the new matrix.

3. Use your calculator and the fact that if **AX** = **B** then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ to solve these systems of equations:

(a) $3x - 2y = 2$
$x + 5y = 29$

(b) $1.2x - 4y = 0.62$
$1.6x - 2.8y = 1.46$

(c) $x + 4y - z = -2$
$3x - y + 3z = 19$
$-2x + y + z = -7$

(d) $3x - z + 4y = 15$
$z - y + x = 0$
$y + 4x - 2z = 17$

4. Consider the effect of the transformation matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on the triangle ABC with A (1,1), B (4,1) and C (4,3). Perform an appropriate matrix multiplication to find the new points A′B′C′ and hence, using a drawing or otherwise, describe the transformation represented by the matrix.

5. Amanda noticed that 4 smoothies and 3 cups of coffee cost \$26, while 2 smoothies and 5 cups of coffee cost \$27. Let the cost of a smoothie be \$*s* and the cost of a cup of coffee \$*c*, set up a pair of equations and use the calculator to solve them to find the costs of each of the drinks.

6. Consider the matrix A below in which each column represents the scores of a group of three students on two different quizzes:

$$\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 2 & 3 \end{bmatrix}$$

Use the transpose command to find **A′A** and verify that the diagonal elements of the resulting matrix are the sums of the squares of the quiz scores while the off-diagonal elements are the cross-products of the quiz scores.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. You are given three transformation matrices $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (a) By considering the effect of each matrix on the triangle ABC with A(-1,1), B(-1,5), C(-4,5), write down the effects of the matrices **P**, **Q** and **R**.

   (b) Show by matrix multiplication that (i) $\mathbf{P}^2 = \mathbf{Q}$ (ii) $\mathbf{P}^3 = \mathbf{R}$ and (iii) $\mathbf{Q}^2 = \mathbf{I}$.
   Explain why these relationships are true.

   (c) Matrices $\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{G} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ all represent reflections.
   (i) Describe the transformation represented by each matrix.
   (ii) Investigate combinations of **F**, **G** and **H** and how combinations of reflections can be equivalent to rotations.

2. Investigate the inverse of a product of two matrices, by choosing some 2 x 2 matrices **A** and **B**. In particular, check to see whether $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$ or $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

3. Investigate the effects of transformation matrices like $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$.

4. Baskets of fruit are prepared for sale at a festival. The *basic* basket has two apples, two peaches and three mangoes. The *deluxe* basket has six apples, three peaches and four mangoes. The *super deluxe* basket has ten apples, five peaches and five mangoes. Altogether there are 420 apples, 310 peaches and 430 mangoes. How many of each type of basket were made up?

5. The determinant of a matrix is needed to find its inverse. If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|\mathbf{A}| = ad - cb$ and $\mathbf{A}^{-1} = \dfrac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Check that matrix $\mathbf{Q} = \begin{bmatrix} 5 & 8 \\ 10 & 16 \end{bmatrix}$ has a zero determinant.
   Explain the significance of this for solving this system of linear equations:

   $$5x + 8y = 6$$
   $$10x + 16y = 3.$$

   Consider some 3 x 3 matrices from this perspective as well.

6. Investigate what happens when a data matrix is pre-multiplied by its own transpose. For example, start with a matrix like **D** below in which each column represents a particular quiz and each row represents a particular student:

   $$\mathbf{D} = \begin{bmatrix} 12 & 14 & 11 \\ 16 & 18 & 17 \\ 7 & 9 & 10 \end{bmatrix}$$

   Study the diagonal and non-diagonal terms of **D′D** carefully. Then try some other data matrices to look for patterns in your results.

Notes for teachers

This module highlights the ways in which the CASIO *ClassWiz* can support students to use matrices. Matrix mode is used throughout the module. The text of the module is intended to be read by students and will help them to see how the calculator can be used to deal with matrices in various ways. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. (a) $\begin{bmatrix} 5 & 10 \\ 15 & -20 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ 4 & -9 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 7 & 10 \\ 1 & -19 & -20 \end{bmatrix}$ (d) Not possible as C is 2 x 2 and D is 3 x 2 (e) $\begin{bmatrix} 5 & 32 \end{bmatrix}$ (f) $\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$ (g) $\begin{bmatrix} \frac{4}{10} & \frac{2}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$ or $\begin{bmatrix} 0.4 & 0.2 \\ 0.3 & -0.1 \end{bmatrix}$ (h) $\begin{bmatrix} -647 & 4320 \\ -1440 & 9433 \end{bmatrix}$ 2. (a) -4 (Use 1 [fraction key] 2 to enter ½) (b) 1 (Use SHIFT 4 and 2 to edit matrix) 3. (a) {4,5} (b) {1.35,0.25} (c) {4,-1,2} (d) {3,1,-2} 4. Clockwise rotation of 90^o about (0,0) 5. Smoothie costs \$3.50 and coffee \$4 6. $\begin{bmatrix} 78 & 92 \\ 92 & 109 \end{bmatrix}$. Properties check: $5^2 + 7^2 + 2^2 = 78$, $6^2 + 8^2 + 3^2 = 109$ and $5 \times 6 + 7 \times 8 + 2 \times 3 = 92$.

Activities

1. The purpose here is for students to explore various transformation matrices. Encourage them to show their results on grid paper and to consider further triangles, not only the one given. [Answers: **P**, **R** are rotations of 90^o about the origin, clockwise and anti-clockwise respectively, while **Q** is a rotation of 180^o about the origin. **F**, **G** and **H** are reflections about the x-axis, y-axis and $y = x$ respectively. **G** followed by **F** = **Q**, etc.]

2. Encourage students to try several examples before leaping to a conclusion. They may be surprised to find that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ rather than $\mathbf{A}^{-1}\mathbf{B}^{-1}$, another reminder about the importance of order of matrix multiplication. You may like to follow up with the general observations like $(\mathbf{B}^{-1}\mathbf{A}^{-1}) \times \mathbf{AB} = \mathbf{B}^{-1}(\mathbf{A}^{-1} \times \mathbf{A})\mathbf{B} = \mathbf{B}^{-1}(\mathbf{I})\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$.

3. Encourage students to experiment with a variety of transformation matrices to see their effects. Plotting results on graph paper will help to understand the effects. Suggest that they try a matrix with a negative stretch factor if this does not occur to them spontaneously. [Answers: the first kind of matrix produces a vertical stretch and the second kind a horizontal stretch.]

4. This activity involves students extracting information from the description and to represent it as a linear system with three variables. The variables are the numbers of baskets not the composition of the baskets. [Answers: 100 basic, 20 deluxe, 10 super deluxe]

5. The definition of a determinant was not given in the text, as it was assumed to be covered in regular teaching. This activity explores a situation giving rise to a zero determinant, because of a linear dependency, and thus the equations have no solution. Encourage students to generate and study more examples of this kind. If they have completed Module 4, they may find it helpful to enter the equations into Equation/Function mode as well.

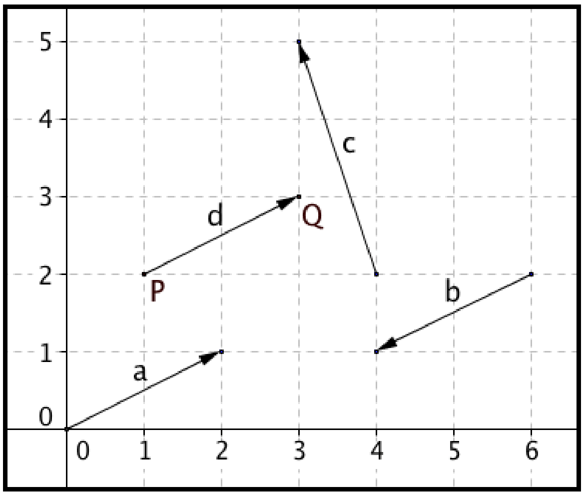
6. Products like **D′D** are important as they are part of the mechanism used by computers to calculate efficiently variance covariance matrices and hence correlation matrices for a set of variables. This activity builds on Exercise 6. Careful study will show why the diagonal elements produce sums of squares and the off-diagonal elements sums of cross-products (thus leading to a symmetric matrix). Encourage students to try another example as well.

Module 8
Vectors

Vectors are used in mathematics, science and engineering where they have an important role in applications of mathematics. The CASIO *ClassWiz* calculator provides significant support for understanding and using vectors. Start this module in Vector mode, by tapping MENU 5.

Representing vectors

The main idea of a vector is that it is a quantity that has both a magnitude (or size) and a direction. Wind speeds and forces are good examples. A common way to represent vectors diagrammatically is using a directed line segment, since this can show both magnitude and direction simultaneously. The computer screen below shows four two-dimensional vectors in a coordinate plane.



There are many ways in mathematics of representing vectors. The computer screen shows that vectors might be represented using letters for the start and end points or simply with a letter as in algebraic notation. Thus, the vector on the screen joining points P and Q can be represented in any of the following ways (and also others) by different people:

$$\overrightarrow{PQ},\ \vec{d},\ \underset{\sim}{d},\ \mathbf{d}$$

In this module, we will generally use a bold lower case letter (such as **a**) to represent a vector, similar to our use of a bold upper case letter (such as **A**) to represent a matrix in Module 7.

In the computer screen, notice that two of the vectors shown (**a** and **d**) have the same direction and are the same length. So, if they were wind speeds, they would represent the same wind strength in the same direction. That is **a** = **d**. The fact that the vectors are shown as starting from different points does not mean that they are different vectors: only the size and the direction are important. Notice that **b** is different from **a** and **d** as it goes in a different direction. Vector **c** has a different size from the other three vectors, and also a different direction.

When vectors are shown on a coordinate screen as above, it is easy to describe them using an ordered pair of numbers. In each case, the two equal vectors show a vector that goes 2 units to the right and one unit up. This is easiest to see when the vector starts at the origin, as **a** does. So **a** might also be described using coordinates in several ways, such as:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \mathbf{a} = [2 \quad 1] \qquad \mathbf{a} = [2,1] \qquad \mathbf{a} = (2,1)$$

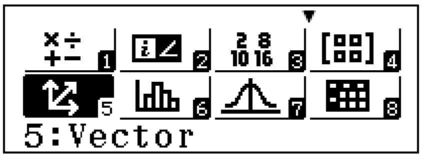
In this module, we will use only the first of these possibilities, to avoid confusion and to be consistent with the calculator representation, as shown below.

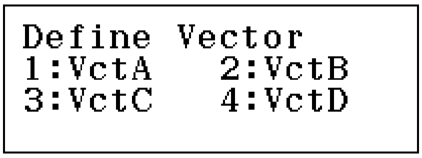
$$\mathbf{a} = \mathbf{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

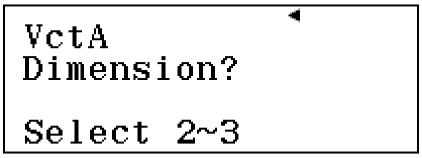
Notice that **d** is *not* named according to the coordinates of its endpoints P(1,2) and Q(3,3), but according to the horizontal and vertical distances 2 and 1 respectively between the end points. Notice that the horizontal component is shown first, and that it is negative if the direction is from right to left, as **b** and **c** illustrate.

It is helpful to think of a vector as similar to a one-dimensional matrix, sometimes called a 'row vector'. (So each of the two vectors above can be thought of as a 2 x 1 matrix.)

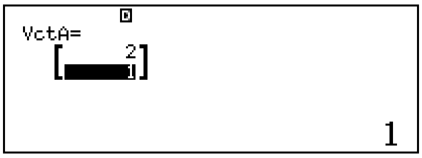
On the calculator, vectors are represented using their numerical components. To define a vector, start by choosing Vector mode with [MENU] [5] and then tap [1] for Vector **a** and [2] to define a 2-dimensional vector.





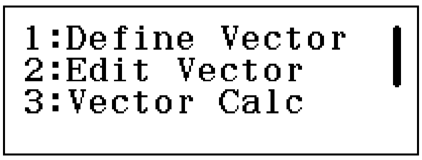


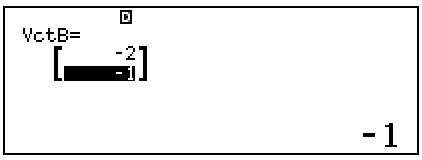
Enter the components, in this case 2 and 1, tapping [=] after each. The screen shows $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

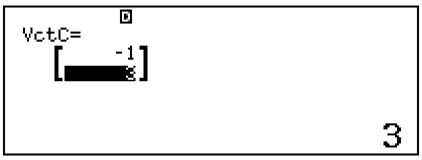


The calculator will allow up to four vectors to be stored. After storing the first one, tap [OPTN] and then [1] to define other vectors.

The screen below shows $\mathbf{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ using these processes.







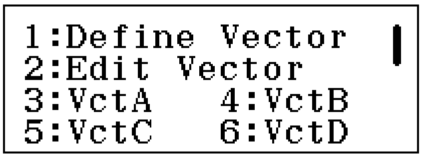
Vector magnitude and direction

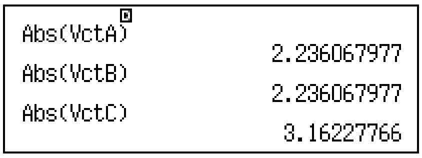
The two defining features of a vector are its magnitude and its direction. For example, in the case of a wind speed, the speed of the wind is the magnitude and the direction towards which it is blowing is the direction. Each of these can be determined on the calculator from the horizontal and vertical components.

The magnitude is represented by the length of the vector. You could determine this by using the Theorem of Pythagoras. For example, the magnitude of **a**, represented by |**a**|, is

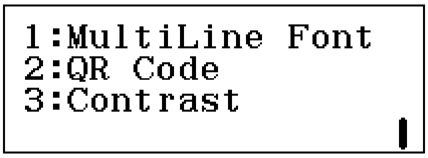
$$|\mathbf{a}| = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.236$$

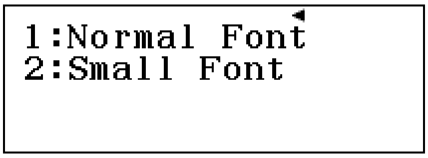
On the calculator, the absolute value key [Abs] (with [SHIFT] [(]) will calculate the magnitude of a vector directly. Clear the screen with [AC], then select [Abs] and select the vectors from the [OPTN] menu. The screen below shows this process repeated for each of the three vectors. The screen shows that **b** has the same magnitude as **a**, even though its direction is different, while the magnitude of **c** is larger (as expected from the diagram on the first page).

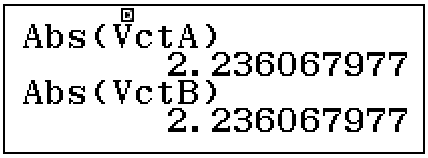




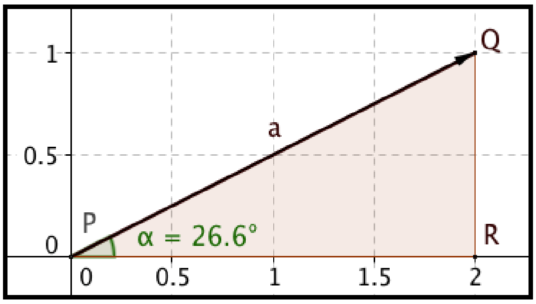
The calculator allows you to show more than one line of information. In this case, all three results above are shown on a single screen, because a smaller font than normal has been selected, using the *Multiline font* command in the fourth SET UP screen. If the Normal font is chosen, only two of the three results fit on a single screen, as shown below. This choice is your personal preference. (In this module we will sometimes show results with *Normal* font and sometimes with *Small* font.)



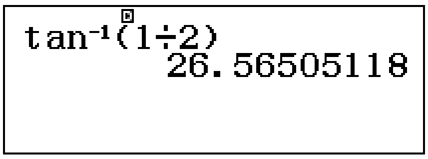


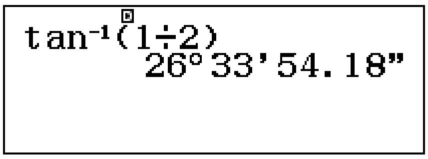


The direction of a vector can be measured by an angle. In this case, we will use the angle the vector makes with the horizontal, measured anticlockwise from the positive *x*-direction. To find this angle, consider a triangle, as shown below and find the angle marked using trigonometry.



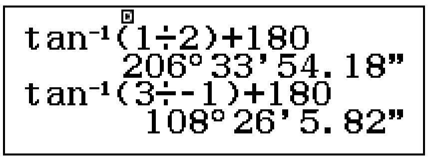
In triangle PQR, $\tan\alpha = \dfrac{QR}{PR}$, so $\alpha = \tan^{-1}\dfrac{QR}{PR}$. This can be calculated using the components of **a**:





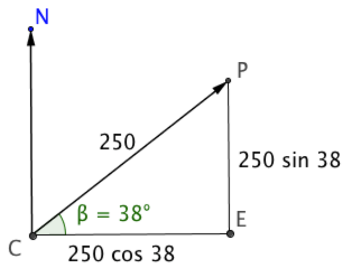
The second screen shows that the [°'"] key can be used to change from decimal degrees to degrees, minutes and seconds (although excessive precision is rarely appropriate).

To determine the direction for **b** add 180^o to that for **a**, since it is a full half-turn further in an anticlockwise direction. The direction for **c** can be obtained in a similar way, but you need to add 180^o to it also, as the calculator gives a negative angle with $\tan^{-1}$ when the tangent is negative.



These two examples make it clear that you need always to draw by hand a rough sketch of vector situations (similar to what we have done at the start of the module) to make sure that the angles obtained are appropriate.

In some situations, you may know the magnitude and direction of a vector, but not know its horizontal and vertical components. For example consider a plane flying North 52° East at a speed of 250 kilometres per hour.

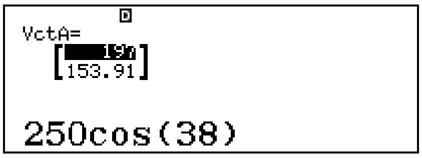


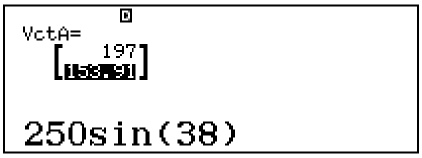
As the diagram shows, the components of the vector $\overrightarrow{CP}$ representing the plane's flight path can be obtained using trigonometry. In the diagram, PE is a vertical line and CE a horizontal line.

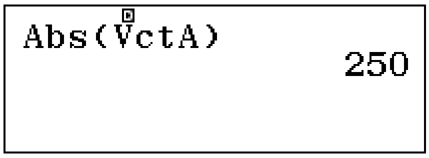
The components in the right triangle PEC are

$$PE = 250 \sin 38^{\circ} \text{ and } CE = 250 \cos 38^{\circ}$$

The vector can be entered as **a** into the calculator using these expressions, as shown below.







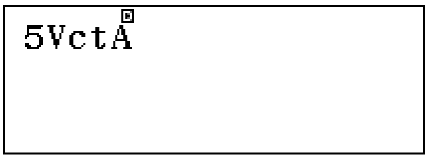
The calculator will evaluate each expression. As a check, note that the magnitude of the vector is 250, as expected.

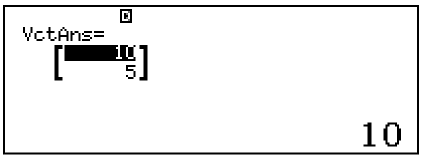
Vector arithmetic

The power of vectors comes from manipulating them, not only from representing them.

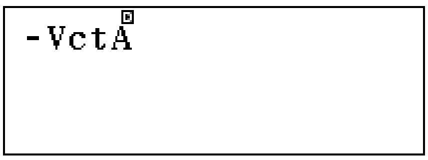
A scalar multiple of a vector is the result of multiplying a vector by a number. Thus the vector 5**a** is a scalar multiple of **a**. As the screen below shows, the effect of doing so is easily seen on the calculator as multiplying each of the components by 5. Tap AC to start a new calculation. Select the vectors as needed from the OPTN menu.

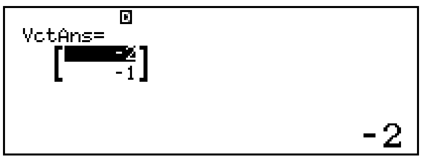
The result of 5**a** is a vector in the same direction as **a**, with 5 times the magnitude, as shown. (The first screen shows the command, and the second screen shows the result of the command.)



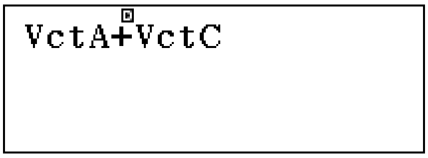


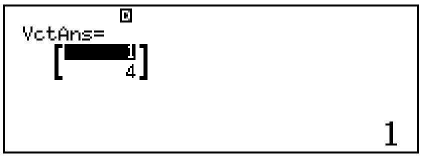
Notice below that multiplying a vector by a negative number reverses the direction (and thus gives the opposite vector.) The screens below show that (-1)**a** = -**a** = **b**.





Vectors can be added, provided they have the same number of components. So we could find the result of **a** + **c** directly on the calculator:



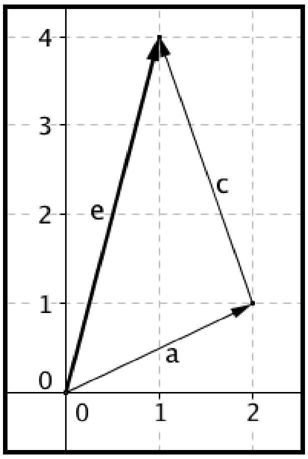


If you consider the two vector components, you can see that the matching components have been added to get the result. Check that 2 + -1 = 1 and 1 + 3 = 4:

$$\mathbf{a} + \mathbf{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

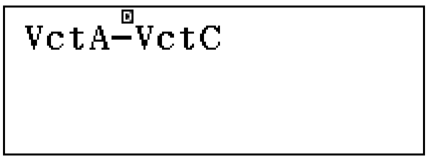
On a diagram, as shown below, the addition of the two vectors results in a third vector, shown in bold, which is the third side of a triangle.

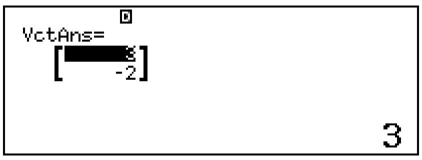
Start with $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then add $\mathbf{c} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ to get $\mathbf{a} + \mathbf{c} = \mathbf{e} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

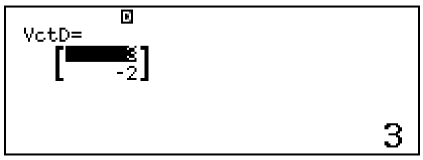


Some people think of this as starting from the origin, and going via **a** and then **c** to reach (1,4) is equivalent to going directly from the origin to (1,4) with **e**.

Subtraction of vectors is related to addition, since to subtract a vector, you need to add its opposite. So, **a** – **c** = **a** + -**c**. On the calculator, subtraction can be performed directly:





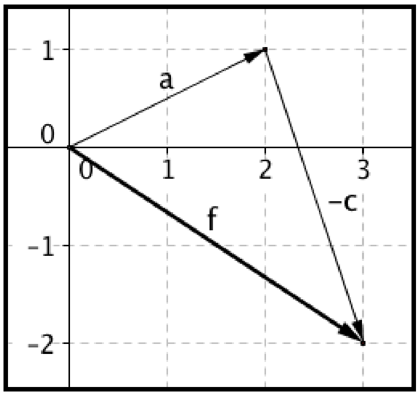


The result of a calculation (named *VctAns*) can be stored into a vector by using the [STO] key, followed by the appropriate letter key (A, B, C or D).

In the right screen above, [STO] [sin] was tapped immediately after the result of the subtraction was obtained to store the result in Vector D. (Neither [ALPHA] nor [=] is needed to complete the storing.) Notice that he calculator screen now shows *VctD* to indicate the storage.

Again, it is instructive to consider this in a diagram.

In this case start with $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and then add $-\mathbf{c} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ (the opposite vector to $\mathbf{c} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$) to get the result $\mathbf{a} - \mathbf{c} = \mathbf{f} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, shown in bold in the diagram.



As you can see from the diagram, the resulting vector of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is the same as that on the calculator.

Notice that $\mathbf{f} + \mathbf{c} = \mathbf{a}$, as expected when $\mathbf{a} - \mathbf{c} = \mathbf{f}$.

An example from sailing

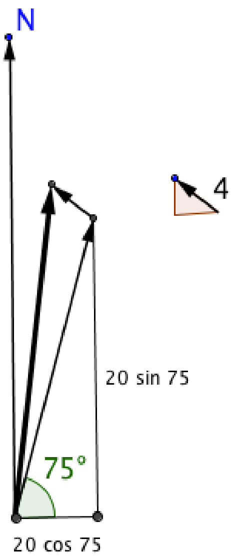
Vector addition is especially useful when two forces of some kind are acting at once. An example is a boat sailing at a particular speed in a particular direction when there is an ocean current affecting it at the same time. Let's look at an example.

> *A fishing boat is sailing in a direction of N15°E at 20 km per hour. There is a current of 4 km per hour in a north westerly direction. Describe the motion of the boat.*

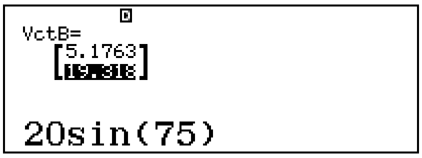
Represent the boat movement and the current movement as vectors **b** and **c** respectively. Then the boat's path, taking both vectors into account, is given by **b** + **c**.

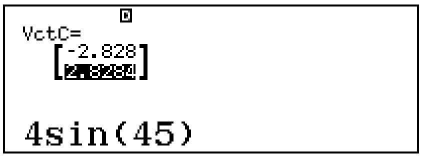
In each case, the vector components need to be calculated. Consider this rough drawing to see that

$\mathbf{b} = \begin{bmatrix} 20\cos 75^o \\ 20\sin 75^o \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -4\cos 45^o \\ 4\sin 45^o \end{bmatrix}$

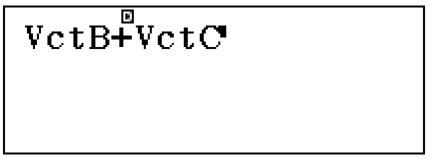


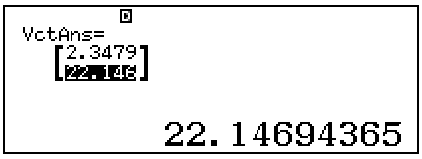
Note the negative sign for the horizontal component of the current.





Enter these into the calculator and find the sum:



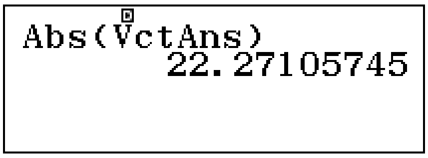


If you wish to study it later, you can store the *VectAns* result as a vector using STO and then the appropriate letter key (for A, B, C or D).

To obtain the direction of the result you will need to write down the components. Rounded to three decimal places, they are:

$$\mathbf{b} + \mathbf{c} = \begin{bmatrix} 2.348 \\ 22.147 \end{bmatrix}$$

Then the boat's speed through the water is given by the magnitude of **b** + **c**:



The boat's direction measured from the horizontal is $\tan^{-1}\frac{22.147}{2.348} \approx 83.95^{o}$.

So the boat is travelling N6^{o}E at 22.3 kilometres per hour.

The effect of the current is to make the boat go a little faster than it would travel in still water and to be pushed a bit closer to a northerly direction, as you might expect from thinking about the situation shown in the diagram (which is not drawn to scale).

Dot product

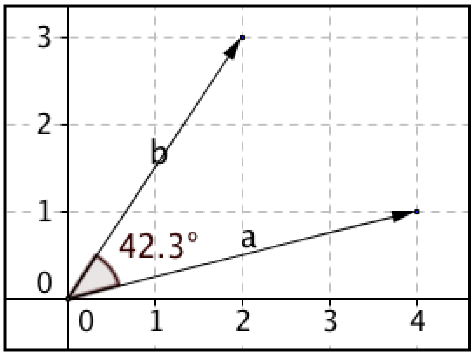
A useful vector operation is the *dot product* of two vectors, sometimes also called the *scalar product*. This is a scalar quantity (i.e. it is a number) defined as:

$$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between the two vectors.

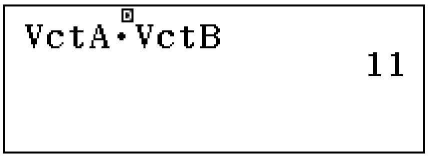
The dot product is sometimes also described as the *inner product* and can be calculated by multiplying the corresponding vector components and adding the products. A calculator operation is available to calculate it automatically.

Consider two vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as shown in the diagram.



Then the dot product is available in the second screen of the OPTN menu by tapping $\boxed{2}$ (Dot Product).

In this case, **a.b** = 4 x 2 + 1 x 3 = 11.



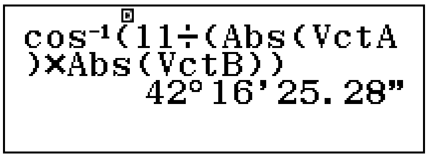
The dot product of two vectors is especially useful as it makes it possible to find the angle between the two vectors, since rewriting the formula above gives

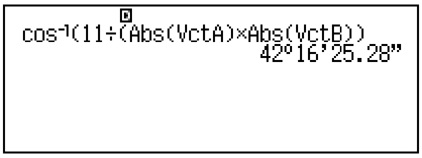
$$\cos\theta = \frac{a \bullet b}{|a||b|}$$

and thus

$$\theta = \cos^{-1}\frac{a \bullet b}{|a||b|}$$

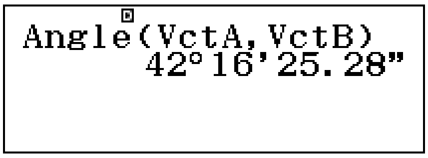
In the case of **a** and **b** above, the angle can be determined once the dot product is found. In the screens below, notice the difference when *Multiline Font* is set to Normal and Small respectively. (The angle size is of course the same in each case.)





The angle looks reasonable, given the drawing above, and is consistent with the automatic calculation above.

In fact, there is a calculator command ($\boxed{3}$ *Angle*) in the second screen of the OPTN menu that finds the angle between two vectors without first evaluating the dot product, as shown below. (



Notice that the dot product will be zero when two vectors are perpendicular to each other, since cos $90^o = 0$. This is sometimes used as a quick test to see whether or not two vectors are perpendicular to each other.

Vectors that are perpendicular to each other are called *orthogonal*.

Three-dimensional vectors

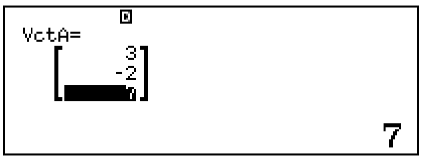
Vectors can also be used to represent quantities in 3D space. While two-dimensional vectors in the plane require two components, three-dimensional vectors in space require three components. Vectors in space are more difficult to draw on this page than vectors in the plane.

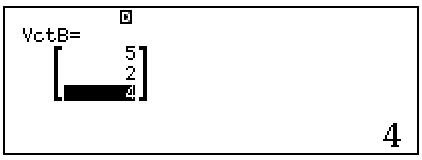
Scalar multiplication, vector addition, vector magnitude and the dot product of two vectors are all calculated in the same way as for two-dimensional vectors.

For example, consider two 3D vectors:

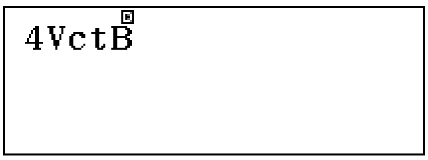
$$\mathbf{a} = \begin{bmatrix}3\\-2\\7\end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix}5\\2\\4\end{bmatrix}.$$

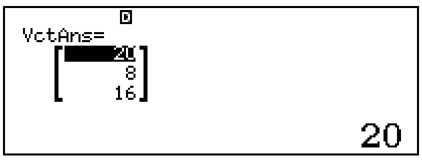
Enter these into the calculator as 3-dimensional vectors:



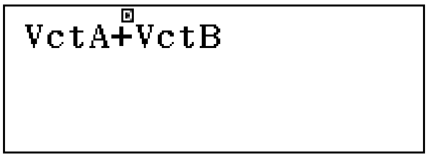


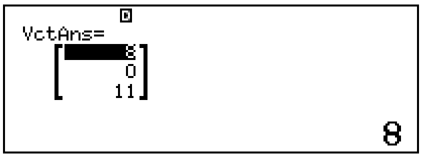
A scalar multiple involves multiplying each component by the same amount to get a vector in the same direction (when the scalar multiple is positive) but of different magnitude. 4**b** is shown here:



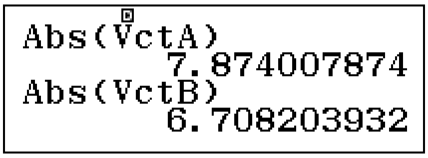


The addition of the two vectors, **a** + **b** = **b** + **a** and is shown here with components added:

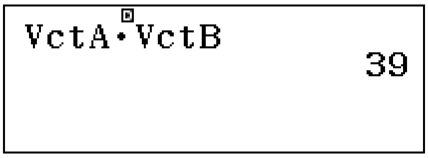




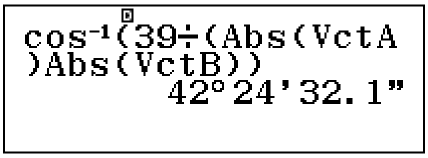
The magnitudes now involve three terms. E.g., |**a**| = $\sqrt{3^2+(-2)^2+7^2}$.

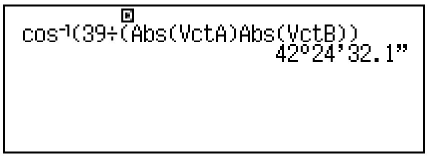


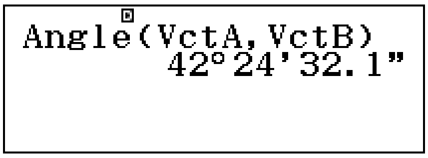
The dot product **a•b** = 3 x 5 + -2 x 2 + 7 x 4 = 39, as shown below:



The angle between the two vectors is available in the same way as for 2D vectors, or with the Angle command.







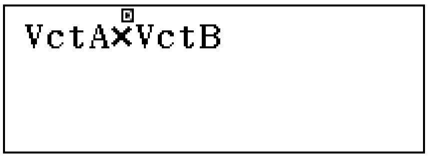
Notice again in the first two screens the consequences of choosing different *Multiline Fonts*.

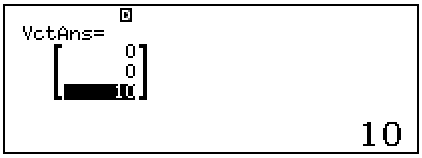
Cross product

Another form of product of two vectors is the *cross product*, sometimes also known as the *vector product*. Unlike the dot product, the cross product of two vectors is itself a vector, that is perpendicular to the plane containing the two vectors.

Let's consider first a two-dimensional example, using **a** = $\begin{bmatrix}4\\1\end{bmatrix}$ and **b** = $\begin{bmatrix}2\\3\end{bmatrix}$, as we used to illustrate the dot product earlier in this module.

The cross product **a** x **b** is obtained on the calculator using the standard calculator multiplication key, as shown below:

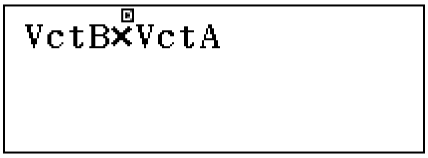


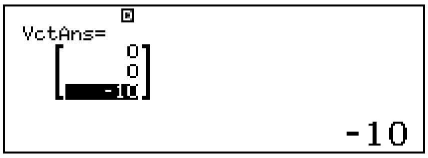


Notice that the result is a three-dimensional vector **a** x **b** = $\begin{bmatrix}0\\0\\10\end{bmatrix}$.

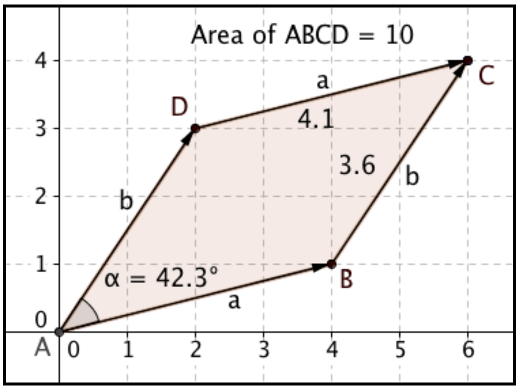
Notice also that the first two components are zero, which tells us that the vector is orthogonal (i.e., perpendicular) to the plane in which vectors **a** and **b** lie. In this case, a three-dimensional vector is needed to obtain a vector perpendicular to the plane.

Unlike dot products, cross products are not commutative. The screens below show that an opposite result is obtained if the order of the vectors is changed:





The magnitude of the cross product is the same as the third component (in this case, 10). The diagram below shows some measurements related to the two vectors:



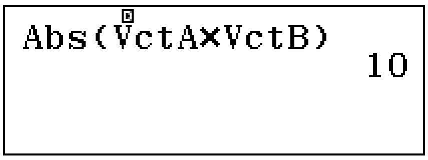
Vectors **a** and **b** are repeated in this diagram to make a shape in the form of a parallelogram ABCD. The area of the triangular shape ABD is given from trigonometry as

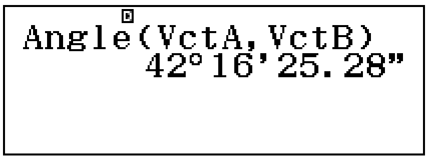
$$\text{Area ABD} = \frac{1}{2}|a||b|\sin\alpha$$

where the angle between the vectors is represented as α. So the area of the entire parallelogram is twice this, or

$$\text{Area of parallelogram ABCD} = |a||b|\sin\alpha$$

As shown on the computer drawing above, this area is equal to the third component of the vector in **a** x **b**. As the first two components are zero, it is also the magnitude of **a** x **b**, as shown below:

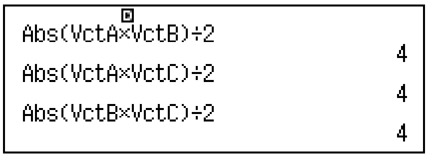


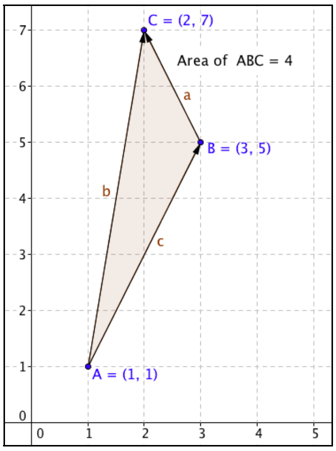


These relationships lead to an easy way to find the area of a triangle in the plane, given the coordinates of its vertices by representing the sides as vectors. Thus if triangle ABC has A(1,1), B(3,5) and C(2,7), then by considering the coordinates, the sides can be represented as vectors:

$$\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The area of the triangle is then half the magnitude of the cross product of any two of these three vectors, as shown below. (The result of 4 is obtained regardless of which pair is used.)

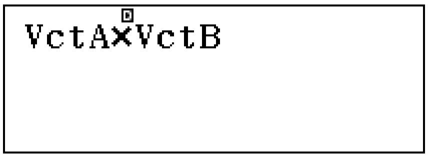


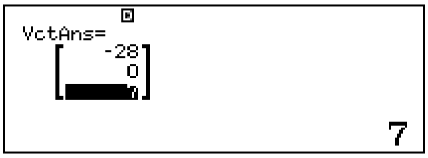


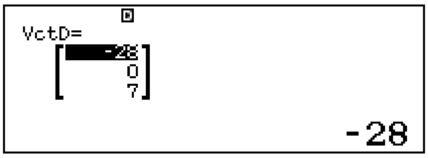
In three dimensions, the cross product will also be a 3D vector, perpendicular to each of the original vectors. For example, consider the two vectors

$$\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

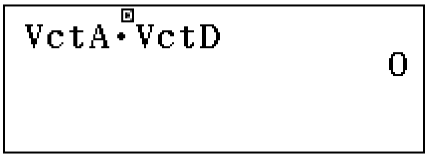
The vector product **a** x **b** is a three-dimensional vector. After the result is obtained, it has been stored as **d** using STO sin, as shown in the third screen below.

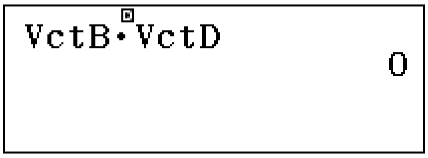




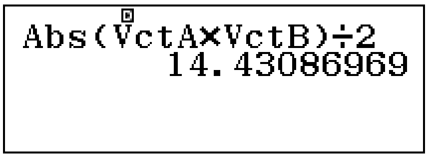


To see that the cross product is perpendicular to each of the original vectors, check that the dot product is zero for each of them. The screens below verify this for each of **a** and **b**.





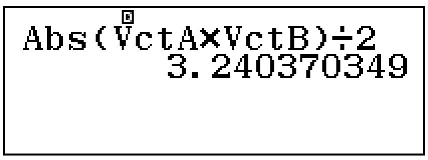
Vectors **a** and **b** are in a plane, which is defined by them, and vector **d** = **a** x **b** is *normal* (or perpendicular) to that plane. As for the two-dimensional case, the cross product can be used to find the area of a triangle with sides represented by **a** and **b**.



Again as for the two-dimensional case, if the vertices of a triangle are known, then the cross-product can be used to find the area of the triangle. For example, if the vertices are A(2,5,4), B(1,6,5) and C(1,1,1) Then vectors for two sides cound be found from the coordinates as:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

The area of the triangle is then given by the following calculation:



Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. Enter $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ into the calculator. Use these to find:

   (a) (i) 3**b** (ii) -2**c** (iii) 10**a** + 2**b**

   (b) (i) |**a**| (ii) |**b**| (iii) |**a** – **c**| (iv) $|\frac{1}{2}\mathbf{a}|$ (v) |-3**c**|

   (c) (i) **a** + **b** (ii) **b** – **c** (iii) 4**c** - **b**

2. Find the acute angle between $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and the horizontal.

3. Find the horizontal and vertical components of a 2-dimensional vector that is 8 units in length and inclined at an angle of 20° to the horizontal.

4. (a) Find |**a**| for $\mathbf{a} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$. (b) Edit the vector to change it to $\mathbf{a} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$. Find |**a**|.

   (c) Change the vector to be a three-dimensional vector $\mathbf{a} = \begin{bmatrix} 5 \\ 12 \\ 0 \end{bmatrix}$. Find |**a**|.

   (d) Edit the vector to change it to $\mathbf{a} = \begin{bmatrix} 5 \\ 0 \\ 12 \end{bmatrix}$. Find |**a**|.

   (e) Explain any differences in the four values for |**a**| found above.

5. Let $\mathbf{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Enter these into your calculator to find:

   (a) **u.v** (b) |**u**| (c) |**v**| (d) The angle θ between **u** and **v**.

6. Kim can swim at 3 km/h in calm water. She swims in a river in which the current flows at 1 km/h in an easterly direction. Find Kim's resultant velocity if she swims (i) with the current (ii) against the current (iii) in a northerly direction.

7. Let $\mathbf{p} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find: (a) **p** x **q** (b) The area of the parallelogram formed from **p** and **q** and vectors equal and parallel to **p** and **q**. (c) the area of the triangle formed by **p** and **q**.

8. Let $\mathbf{u} = \begin{bmatrix} 1 & -4 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 & 7 & -1 \end{bmatrix}$ be three-dimensional vectors.

   Find: (a) 5**u** + 3**v** (b) |**u**| (c) **u.v** (d) **u** x **v** (e) the area of a parallelogram for which these two vectors represent two sides (f) the angle between the two vectors

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. A quadrilateral PQRS has vertices P(3,2), Q(4,5), R(9,6) and S(8,3).
   (a) Use vectors to describe the four sides $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{SR}$ and $\overrightarrow{PS}$.
   (b) Use your result from (a) to decide what sort of quadrilateral PQRS is.
   (c) Find the area of the quadrilateral PQRS and the triangle PQR.

2. Let $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

   (a) Find **p.q**, **q.p**, **p.r** and **r.p** (b) Find **p.p**, $|\mathbf{p}|^2$, **q.q** and $|\mathbf{q}|^2$ (c) Find **p.(q + r)**, **p.q** + **p.r**, **q.(p + r)** and **q.p** + **q.r** (d) What properties of dot products are suggested by your answers to parts (a), (b) and (c)? Check these properties with some three-dimensional vectors of your own choosing.

3. Two tugs are used to pull a barge in a water festival. The tow ropes of equal length are attached to a single point on the barge and are 32° apart. If each tug pulls with a force of 2500 newtons, find the actual combined pulling force.

4. (a) A light aircraft is flying at 300 km/h aiming due south. Find its actual speed and direction if there is a 40 km/h wind from the south-east.

   (b) A charter boat needs to go as quickly as possible from a fishing location to its mooring in the harbour, which is 65 km away and directly north. If the top speed of the jet-propelled boat is 80 km/h and there is a current flowing at 10 km/h from the north-east, in which direction should the captain chart a course? How long will it take for the boat to arrive?

5. Consider triangle ABC with A(1,1), B(5,-1) C(4,-3). Use the scalar (dot) product to check if the triangle is right-angled. Find the area of the triangle.

6. (a) Find three vectors that are perpendicular to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

   (b) Find the general form of all the vectors that are perpendicular to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

   (c) Describe the vectors that are perpendicular to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

   (d) Find the general form of vectors that are perpendicular to these vectors:
   (i) $\begin{bmatrix} 3 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} -3 & -5 \end{bmatrix}$

   (e) Explore questions like those in parts (a) to (d) in three dimensions. Start by finding vectors that are perpendicular to $\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$.

Notes for teachers

This module highlights the ways in which the CASIO *ClassWiz* can support students to think about vectors and use them in applications of mathematics. The module makes extensive use of Vector mode and uses a notation for vectors consistent with the calculator. The text of the module is intended to be read by students and will help them to see how the calculator can be used to examine various aspects of vectors. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. (a) $\begin{bmatrix} 3 \\ 12 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \end{bmatrix}, \begin{bmatrix} 22 \\ -2 \end{bmatrix}$ (b) $\sqrt{5} \approx 2.24$, $\sqrt{17} \approx 4.12$, $\sqrt{61} \approx 7.81$, $\sqrt{1.25} \approx 1.03$, $\sqrt{306} \approx 17.49$ (c) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -13 \\ 16 \end{bmatrix}$ 2. 36.87^o 3. $8\cos 20^o \approx 7.52$ and $8\sin 20^o \approx 2.74$ 4. All lengths are 13 5.(a) 21 5.(b) 5 (c) $\sqrt{37} \approx 6.08$ (d) 46.33^o 6. (a) 4 kph (b) 2 kph (c) 3.2 kph direction N18.4°E 7. (a) $\begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$ (b) 17 (c) 8.5 8.(a) $\begin{bmatrix} 11 \\ 1 \\ 7 \end{bmatrix}$ (b) 4.583 (c) 7.348 (d) -28 (e) $\begin{bmatrix} -10 \\ 5 \\ 15 \end{bmatrix}$ (f) 18.71 (g) 146.25^o

Activities

1. Students should recognise that equal vectors for the pairs of sides indicate that the shape is a parallelogram, so the cross product gives its area and the area of the triangle is half that of the parallelogram. [Answer: areas are 14 and 7.]

2. The chosen vectors are used here to explore properties of dot products. The activity illustrates that the operation of a dot product is commutative and distributive over vector addition and that the dot product of a vector with itself is the square of its length. Students should be encouraged to check with other 2D and 3D vectors as well to see for themselves that the results hold generally. [Answers: (a) 5, -11 (b) 5, 25 (c) -6, -6, -12, -12 (d) as above.]

3. For these kinds of applications considering the combined effects of two vectors, drawing a diagram is essential. Students should use their diagrams to see that the forward force for each tug is $2500 \cos 16^o$, so the combined force is the sum. [Answer: 4806.3 newtons]

4. Part (a) requires the addition of two vectors for the plane and the wind, while part (b) requires vectors for the course to be charted, the boat's heading and the current. Encourage students to draw diagrams. [Answer: (a) plane is travelling at 273.2 kph in the direction S5.9°W (b) Boat heads N16.87°W at a speed of 53.33 kph, and takes about 1 hour 13 minutes]

5. This activity exploits the property that orthogonal vectors have a zero dot product. After writing the triangle sides as three vectors, $\overrightarrow{AB} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \overrightarrow{CA} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, students should find that $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$, so the triangle is right angled at B with area of ½ |AB||BC| = 5 square units.

6. This activity also uses the property that orthogonal vectors have a zero dot product and generalises the result. [Answer: (a)-(c) perpendicular vectors are scalar multiples of $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (d) scalar multiples of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ (e) The general equation $3x + y + 6z = 0$ defines a plane through the origin; $\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$ is a normal to the plane.]

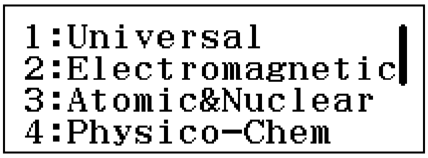
Module 9
Further numbers

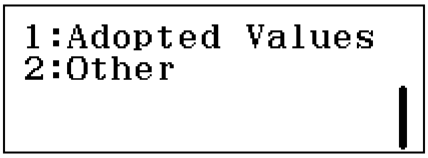
The CASIO *ClassWiz* calculator deals with fractions, decimals and percentages, as shown in Module 2, and also uses scientific notation and engineering notation when necessary or desired. In this module, we will explore some of the other aspects of numbers handled by the calculator. These are of less general and more specific interest, so you should use those of particular interest to your study or work.

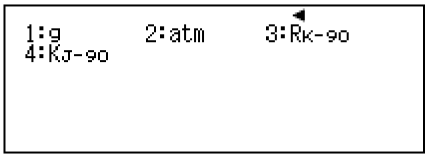
Scientific constants

Especially in the physical sciences and engineering, a number of constants are in regular use. For convenience, a careful selection of 47 of these are available for quick retrieval and use. There is a list of the standard symbols for these in the User's Guide, but you will probably recognise those that are familiar to you.

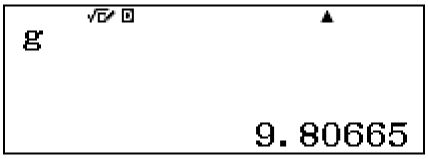
These constants can be recalled and used in calculations when appropriate using the CONST command (SHIFT 7). The constants are grouped into areas of application, as shown below. The example on the right shows the Adopted Values.







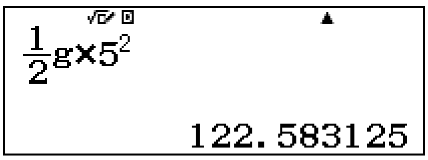
For example, the acceleration due to gravity at the surface of the Earth is represented by *g*. It is an important constant in various calculations related to falling objects. The accepted value of the constant can be retrieved by selecting it from the Adopted Values screen with 1 and tapping =, as shown below. It is not necessary to recall the value in this way to use it in a calculation, however.



The velocity *v* of a falling object *t* seconds after being dropped from rest to the Earth is given by:

$$v = \frac{1}{2}gt^2$$

So, to determine the velocity of a parachutist after five seconds of free fall, the calculator can be used as shown here, with the symbol for the constant *g* retrieved as above.

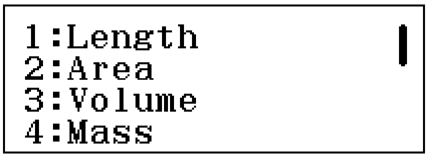


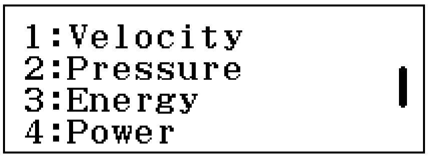
The resulting velocity is 122.6 metres per second (as the units are assumed to be metres and seconds here, as is generally the case in the physical sciences). Notice that the number itself does not appear in the calculation – only the symbol *g* for the constant. Notice also that it is necessary in this case to include the multiplication sign in the calculation.

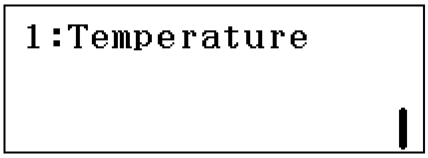
The constants in the calculator are based on the 2010 work of the Committee on Data for Science and Technology (CODATA) for international use. Some constants are periodically revised (slightly), as scientists get better measurements of them, so you should consult the Internet if you wish to be aware of these details.

Measurement conversions

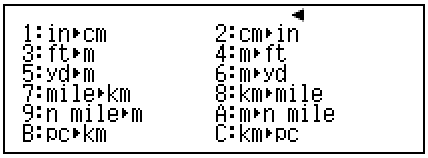
Most measurements in the world of science and technology use the metric system, but some measurements are still reported in other systems in some countries. To convert between systems for commonly used units, the calculator has a set of 20 pairs of unit conversions. These are grouped into nine measurement areas, as shown below with the SHIFT 8 (CONV) command, and the ▼ and ▲ cursors. (Notice that you can use ▲ to go up to the final screen from the first screen.)



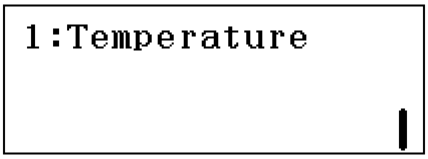


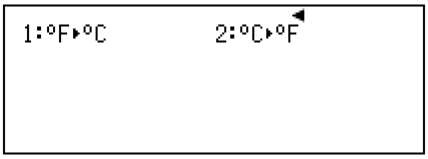


The most extensive conversion pairs are for length, shown below after tapping 1: Length. Conversions are self-explanatory, using standard symbols for the units concerned. Notice (from the small ◀ symbol in the top bar) that you can return to the measurement groups with ◀ if the conversion you want is not shown.

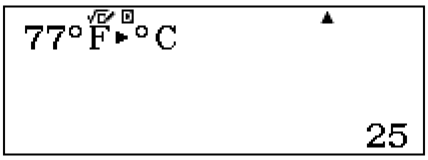


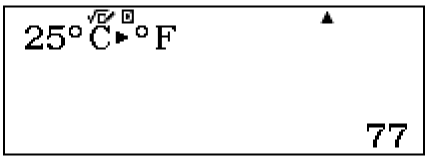
Similarly, conversion between temperatures measured in Celsius and Fahrenheit are available in the Temperature group, shown at left below, after tapping 1.





To convert a quantity on the calculator screen from one measurement to another, first enter the quantity to be converted, then select the appropriate conversion. For example, to convert a temperature from 77°F to Celsius, enter 77 (start by clearing the screen with AC, if the conversion factors are still showing). Then select the appropriate conversion and tap =. The first screen below shows this, while the second screen shows the conversion from 25°C back to 77°F.



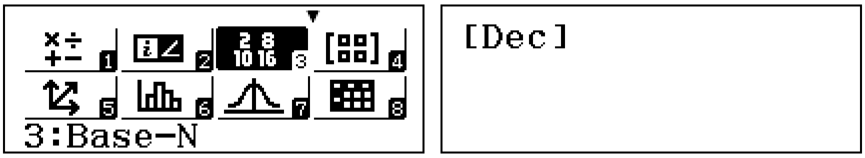


In this particular case the calculator is using the relationship that $F = \frac{9}{5}C + 32$ in both directions.

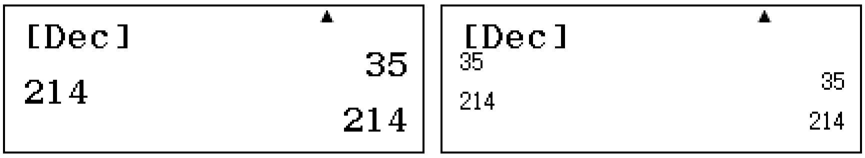
Conversions are based on the work of the National Institute of Standards and Technology (NIST) in the USA, and you can refer to the Internet for further detailed information. Conversion factors for the units in the calculator are unlikely to change at the level of precision used, and you can use them with confidence without referring to the NIST for updates.

Numbers to other bases

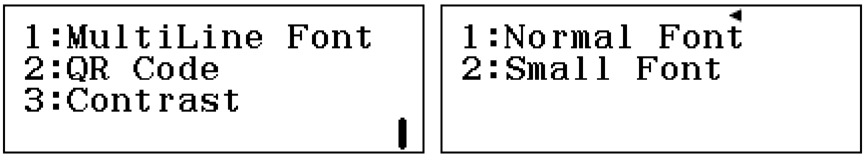
The decimal number system is used throughout the world, and so is the basic system used to represent numbers in the calculator, as explored in Module 2. However, for some purposes, most notably in computer science, other number bases are sometimes used. The decimal system uses base 10, while other systems use base 2 (binary), base 8 (octal) and base 16 (hexadecimal). The calculator allows for conversions between positive whole numbers in these systems. To use this feature, set the calculator to Base-N mode, using MENU 3. The calculator will default to Decimal mode, as the screen below indicates.



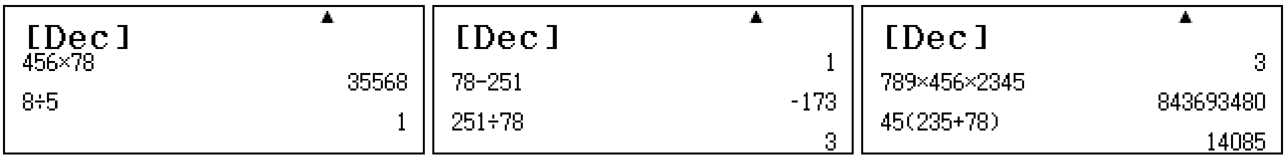
If you enter whole numbers (such as 35 or 214) and tap =, they will be assumed to be decimal numbers, as shown below.



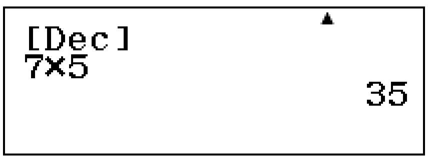
The two screens above show that the calculator can be set to use either a Small or Normal *Multiline Font* in this mode, through the use of SET UP (SHIFT MENU). A Small font will sometimes allow you to see more lines at once, as the second screen above shows. Changes will take effect immediately. Choose whichever you prefer. (In this module, both sizes are used at various times.)



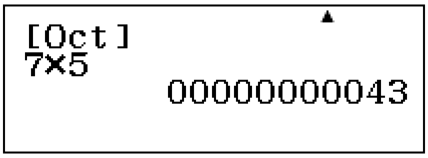
Only whole numbers can be used in Base-N mode. Other keys, such as the decimal point and square root keys will not function as normal, and will be ignored. Operations of +, −, × and ÷ can be used, but only with whole number results (so decimal base divisions will not result in decimals, for example). Similarly, large numbers will give an error, rather than be represented in scientific notation, as in other modes. Study the screens below to see examples, and try some like these for yourself in DEC mode.

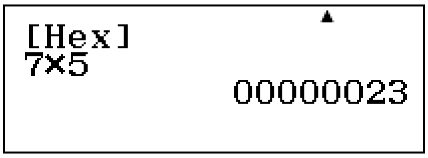


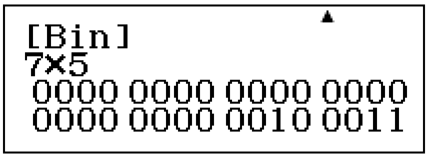
To convert a number from one base to another, note the DEC, HEX, BIN and OCT commands written in blue above the x^2 $x^■$ log■□ and ln keys. If you tap any of these keys, the results of calculations in the display will be represented accordingly in the chosen base. To see how this works, clear the screen with AC and complete a calculation such as that below.



Then tap the OCT, HEX and BIN keys in succession, to see the result to the three different bases:





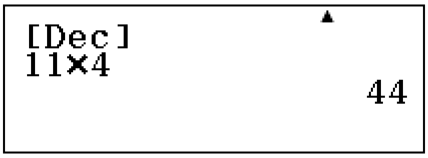


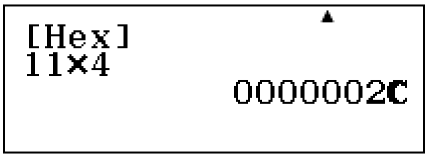
Study these screens carefully. Ignore all the leading zeroes. Since octal numbers use base 8 instead of base 10, then 43_8 in the first screen means 4x8 + 3 = 35 in base 10. Hexadecimal numbers use base 16, so the second screen shows 23_{16} = 2x16 + 3 = 35 in base 10. Binary numbers use base 2, so that the third screen shows that 100011_2 = 1x32 + 0x16 + 0x8 + 0x4 + 1x2 + 1 = 35 in base 10. That is, the results shown above are all the same number, 35_{10}, represented in different bases.

Notice that the calculation that was entered (7 x 5 in this case) is unaffected by these changes and is still shown on the screen in each case.

When a number system is chosen using one of the four blue keys, you will only be able to enter numbers that the system understands. Thus, entering any digits except 0 and 1 in binary will result in a Syntax Error. Similarly, 8 and 9 cannot be used in octal (in the same way that there is no symbol for ten in decimal).

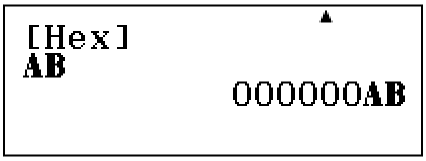
The hexadecimal system, in contrast, requires *extra* symbols (to represent the numbers from 10 to 15), as is clear sometimes when numbers are converted into hexadecimal, as below:

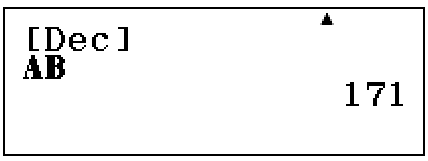


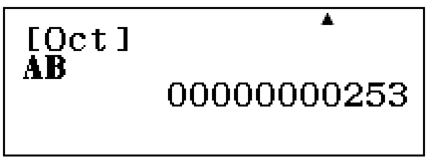


In hexadecimal, the decimal number 11 x 4 = 44 is represented as 2C. The digit C refers to 12 in base 10, so $2C_{16}$ = 2x16 + 12 = 44 in base 10.

To enter hexadecimal numbers in the calculator, use the row of red letter keys (A to F) usually used for variable memories but which in this mode are restricted to that use. In order across the calculator, (in base 10), A = 10, B = 11, C = 12, D = 13, E = 14 and F = 15. Notice that these keys can only be used when the calculator is set to hexadecimal, or a Syntax Error will result. Here is an example of converting from the number AB_{16} to decimal and to octal. Set the calculator to hexadecimal mode (using $x^{\blacksquare}$) and then enter AB (using (−) °,,,) and tap =. Tap x^2 and ln for the conversions.

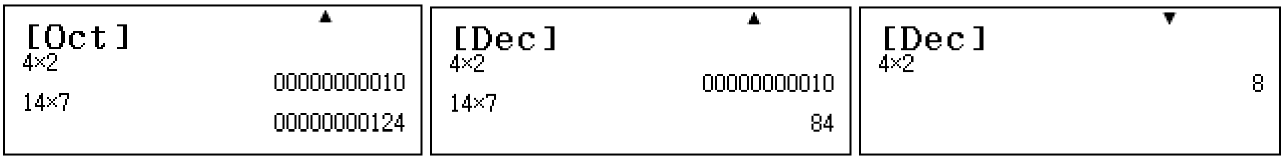




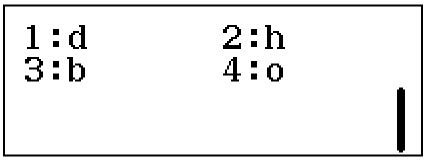


Notice that AB_{16} = 10x16 + 11 = 171 in base 10. Similarly, 253_8 = 2x64 + 5x8 + 3 = 171 in base 10. Notice again that the original hexadecimal entry AB is left on the screen.

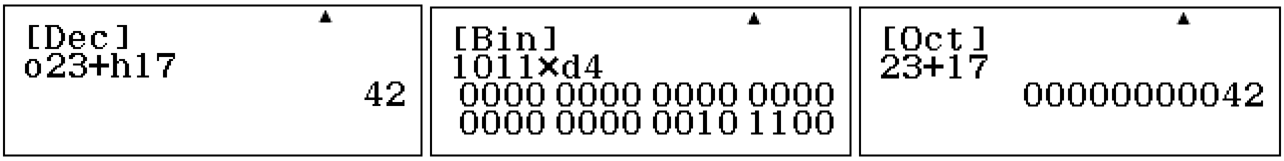
If you have undertaken a series of calculations, recall that the small ▲ in the screen will indicate that you can retrieve these. Changing the base will affect only the most recent calculation, but you can change the base of previously entered calculations in turn using ▲. Study the following screens carefully to see how this works. Two calculations were entered in octal mode, and then converted to decimal with DEC (x^2). The first conversion, shown in the middle screen, was made immediately, but the second conversion was only made after ▲ was tapped.



There is another way to enter numbers in different bases in the Base-N mode, using the OPTN menu. Tap OPTN ▼ to enter the second screen to see the four prefixes shown below.



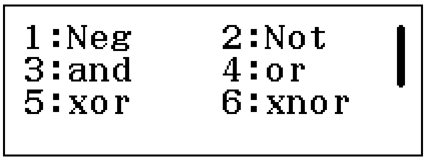
If numbers are entered with a prefix, numbers in different bases can be shown in the same screen. The first screen below shows the sum of 23_8 and 17_{16} in decimal mode. The second screen shows that multiplying a binary number by 4 can be achieved by adding two zeroes after the number. (Why?) Notice that the prefix must precede the number.



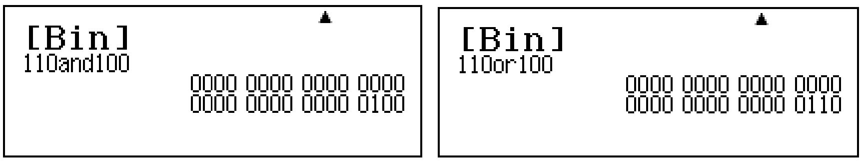
When numbers do not have any prefix, they are assumed to be in the present base. Thus the right screen above in octal base shows $23_8 + 17_8$. Check for yourself that each of these two additions is correct.

Binary logical operations

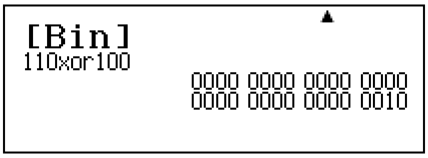
The OPTN menu in Base-N mode also contains various logical operations, used for work with binary numbers. So, in this section, set to binary numbers with the blue log key. The opening OPTN menu screen is shown below.

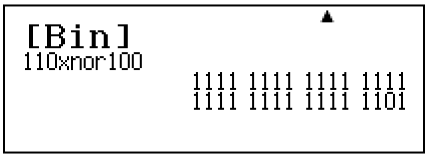


These are used in computer science to perform operations comparing binary numbers. For example, the *and* operation produces a binary number that is 1 if the matching binary digits are both 1s and produces a 0 otherwise. For 110_2 and 100_2, only the first digit is 1 in each case. The *or* operation produces a 1 if *either or both* of the two numbers has a 1 in a particular place. For 110_2 and 100_2, this is the case for the first two digits only. The screens below show these commands in use.

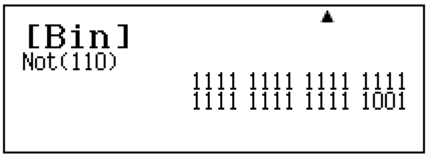


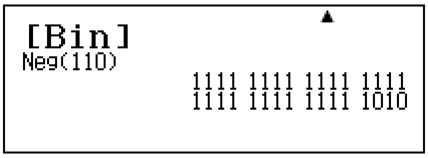
As for mathematics in general, the *or* command is an *inclusive or* (one or the other or both). The *xor* command is an *exclusive or*, meaning one or the other but *not* both. For 110_2 and 100_2, only the second digits meet this criterion. The *xnor* operation is the opposite of this, as shown below, as it returns a 1 whenever the matching digits of a pair of numbers are the same and a 0 whenever they are different. (Note the last three digits of the second screen, 101, in particular)





Finally, the two negation commands, *Not* and *Neg*, apply to individual binary numbers. The Not command switches 1 and 0 digits in the number. The Negative command gives the 'two's complement' of a number, used in designing computer operations. (You can see from the screens below that the Negation of a number is one more than the value given by a Not command.)





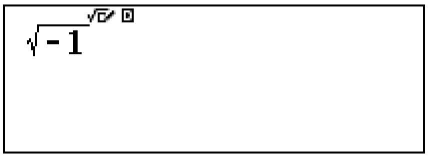
Of course, these operations are likely to be of interest only to those working extensively with binary numbers, especially those in computer science domains.

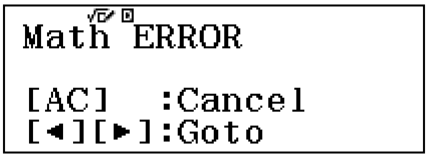
Complex numbers

Complex numbers are important in many parts of advanced mathematics, and you may have already noticed them appearing automatically in Module 4. Sometimes they are called 'imaginary' numbers, although they are no more imaginary than other numbers. The most basic complex number is the square root of negative 1, represented by ***i***.

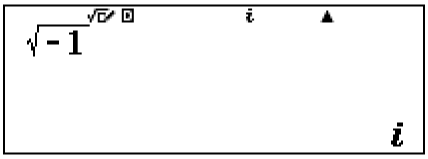
$$\sqrt{-1} = i$$

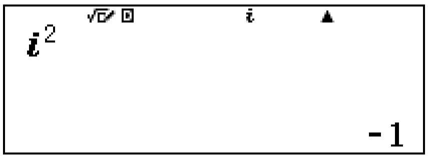
If you try to determine this on the calculator in Calculation mode, however, an error will occur:





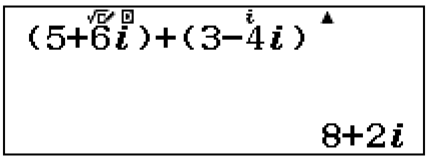
However, if you switch the calculator to Complex mode with MODE 2, there will not be an error:

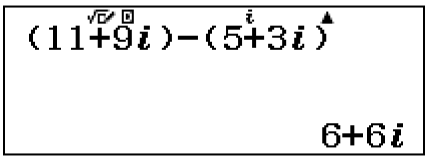




Notice that the calculator display shows a small ***i*** at the top to indicate that Complex mode is in use. (You may have seen this symbol in Module 4 when solving polynomial equations.) When in Complex mode, ***i*** itself can be obtained by tapping the ENG key, as indicated by the small purple ***i*** symbol above the key. (It is *not* necessary to tap the SHIFT key first). The fundamental property of ***i*** is that its square is -1, as shown above on the calculator.

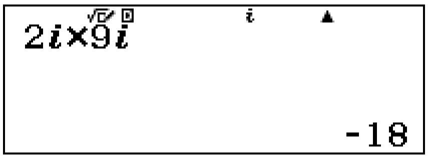
Complex arithmetic can be completed on the calculator, although some calculator functions will not work with complex numbers. When complex numbers are added and subtracted, the screens below show that the real parts and the imaginary parts are handled separately to get the final result.

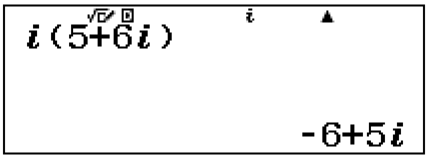




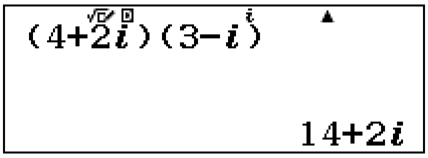
Notice that *ClassWiz* results always display the real part of complex numbers first.

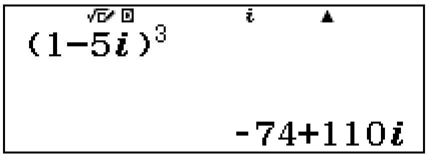
Multiplication by $\boldsymbol{i}$ or a multiple of $\boldsymbol{i}$ relies on the property that $\boldsymbol{i}^2 = -1$, as shown below:

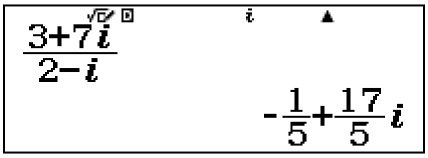




Here are some further examples, showing how multiplication and division work:





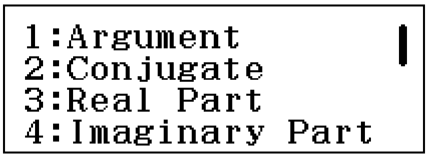


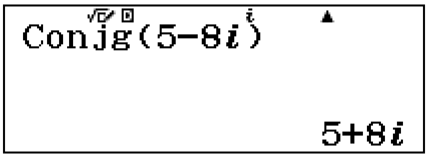
Notice in the third case that the denominator has been automatically rationalised (so that it has no complex numbers in the denominator). To see how the calculator has done this, study below how the division can be expressed to avoid any complex numbers in the denominator:

$$\frac{3+7i}{2-i} = \frac{3+7i}{2-i} \times \frac{2+i}{2+i} = \frac{-1+17i}{4-i^2} = \frac{-1+17i}{5}$$

This process relies on multiplying $2 - \boldsymbol{i}$ by $2 + \boldsymbol{i}$ in the denominator; the real part of the number is unchanged but the imaginary part has been reversed. $2 + \boldsymbol{i}$ is described as the *conjugate* of $2 - \boldsymbol{i}$.

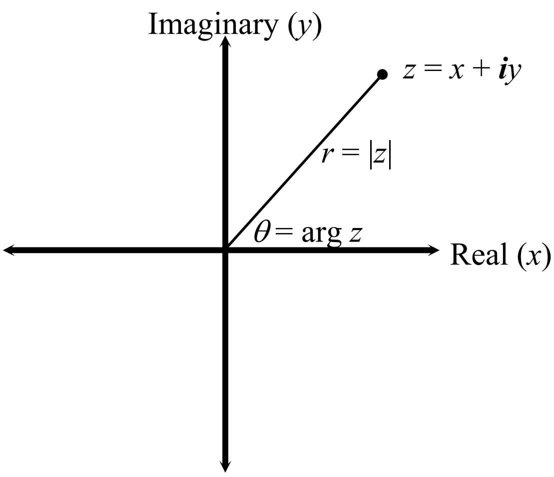
Special complex commands are available in Complex mode via the OPTN menu. The screen below shows how the conjugate of a complex number can be found (although it is often just as easy to do this mentally if only one complex number is involved)





Argand diagrams

Complex numbers are often represented graphically. While a real number can be represented as a point on the number line, a complex number $z = x + \boldsymbol{i}y$ can be represented as a point on the *complex plane*. This plane has a real axis (x) and an imaginary axis (y). A diagram representing complex numbers in this way is called an *Argand* diagram.

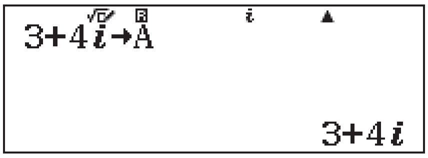


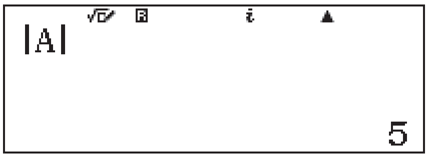
The Argand diagram also shows the *modulus* or *absolute value* of the complex number, represented by $r = |z|$, which you can think of geometrically as its distance from the origin. The Theorem of Pythagoras will help you to see that

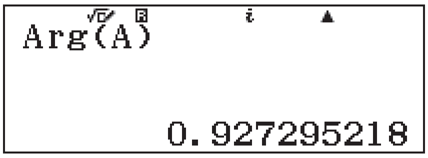
$$r = |z| = \sqrt{x^2 + y^2}\,.$$

The angle made at the origin with the positive real axis is called the *argument* of the number, arg z.

Although you can determine these values from the real and imaginary components of a complex number, it is easier to use the inbuilt functions in the [OPTN] menu. In the screens below, the calculator has been set to radians and the complex number $a = 3 + 4\boldsymbol{i}$ has been saved to memory A for convenience.







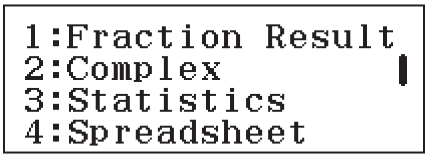
The modulus function is the same as the absolute value function (*Abs*) for real numbers, obtained with [SHIFT] [(]. The argument function is available through the [OPTN] menu.

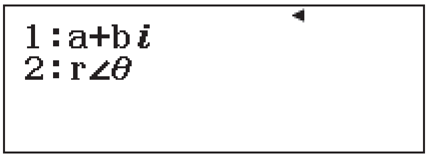
Thinking about complex numbers geometrically will often be helpful. For example, the conjugate of a complex number is the reflection of the number in the real axis.

Polar form

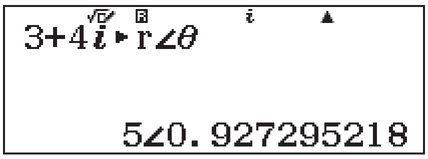
Because of the geometric interpretations, complex numbers are sometimes represented in polar form, giving their distance from the origin and their angle with the positive real axis, the argument of the number. So, the complex number $a = 3 + 4\boldsymbol{i}$ can be represented in coordinate form as (3,4) or in polar form as [5,0.9273]. Polar coordinates are represented with square brackets to avoid confusion with rectangular coordinates.

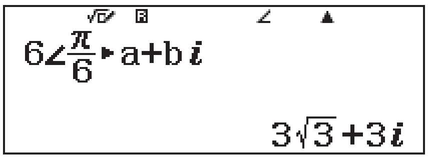
Because there are two possible representations of complex numbers, you can choose which you want to use with the SET UP ([SHIFT] [MENU]) menu (on the second page). Tap [2] and choose one of the two forms.





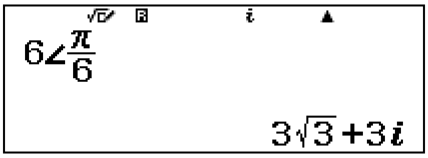
The calculator allows you to convert between these two forms, using the second page of the [OPTN] menu. Make sure that the calculator is set to give results in Math mode, using SET UP if necessary to do this. The screens below show both possibilities.





When entering numbers in polar form into the calculator, you need to access the argument (angle) symbol with [SHIFT] [ENG] (∠), used after the modulus and before the argument.

At present, our calculator is set to show complex numbers in coordinate form as a + b**i**. So, if a result is obtained, or a number entered, it will automatically be represented in this form, as shown in the first screen below, and a conversion command will not be needed.

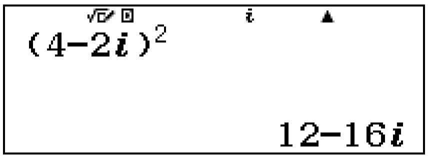


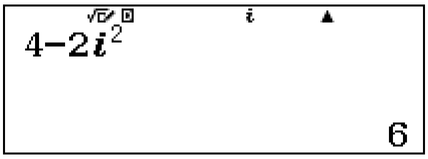
Check for yourself that if the calculator is set to show complex numbers in polar form then entering $3\sqrt{3} + 3\boldsymbol{i}$ will result in the polar form above being displayed, as in the second screen.

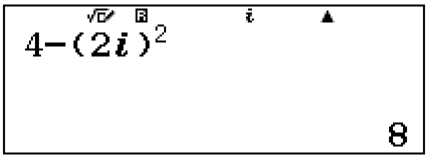
Powers and roots of complex numbers

You may have already noticed that you can use the square x^2, cube x^3 and reciprocal x^{-1} keys with complex numbers, but not the root $\sqrt{\blacksquare}$ or the power $x^{\blacksquare}$ keys (except for whole number powers). This is because the calculator uses multiplication and division internally for these particular powers, but a different process (using logarithms) for roots and other powers; since logarithms of complex numbers are not defined, a Math error will result if $\sqrt{\blacksquare}$ or $x^{\blacksquare}$ keys are used.

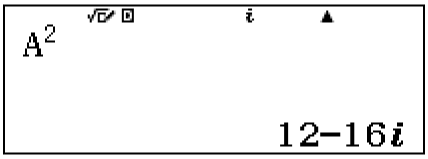
When using the powering keys, be careful how you enter the expressions. Study the following three screens carefully. The first screen below shows the correct way to square the number $4 - 2\boldsymbol{i}$. The second screen shows that tapping the x^2 key immediately after the number squares only the $\boldsymbol{i}$ in the complex part of the number, and is thus incorrect. Similarly, the squaring in the third screen, applies only to the complex part of the number.







In the screen below, the complex number $4 - 2\boldsymbol{i}$ has already been stored in memory A, and so can be squared with the x^2 key.

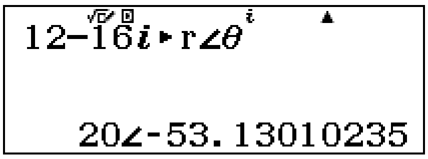


To obtain fractional powers or roots of complex numbers, you will need to use the remarkable result called *De Moivre's Theorem*: for a complex number in polar form, $[r, \theta]$,

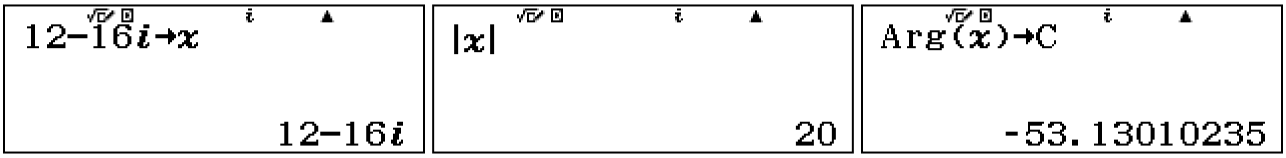
$$[r,\theta]^n = [r^n, n\theta] .$$

For example, you can use this result with $n = ½$ to find a square root of a complex number $12 - 16\boldsymbol{i}$.

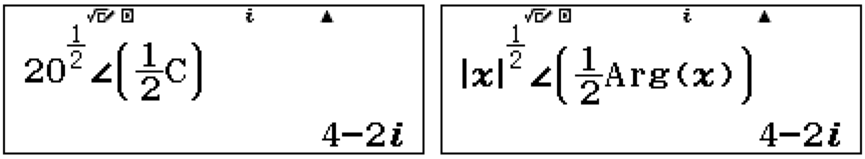
First it is necessary to express the number in polar form. Although the command below achieves this, it is not very helpful for the rest of the computation (because it requires recording the argument, and thus possibly losing accuracy):



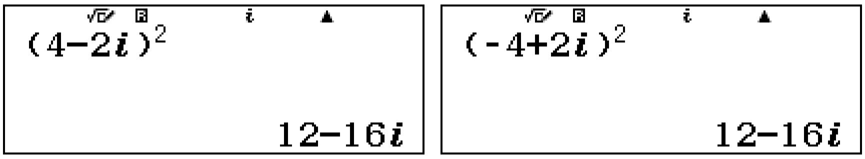
So, a better approach is to find the modulus (r) and the argument (θ) separately, as shown in the following screens. To save some steps, first store the number in memory x. It is wise to store the non-integral result for the argument into a memory to retain all the accuracy available.



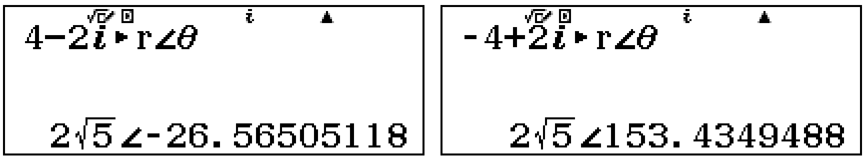
Then a square root is available directly via de Moivre's Theorem. The first screen below uses the values just obtained, while the second screen shows that it was not really necessary to obtain and display these values in order to use them:



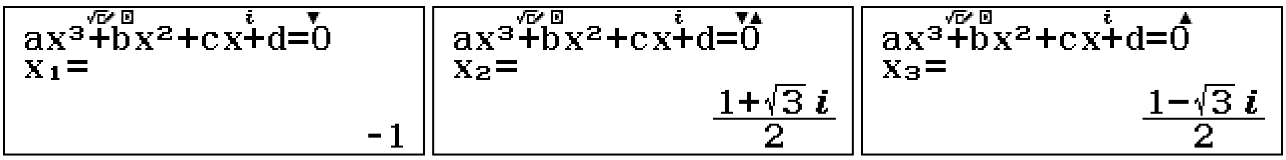
The other square root is obtained by finding (mentally) the opposite of the result: -(4 – 2***i***) = -4 + 2***i***. You can check that these are in fact square roots of 12 – 16***i*** by squaring the results:



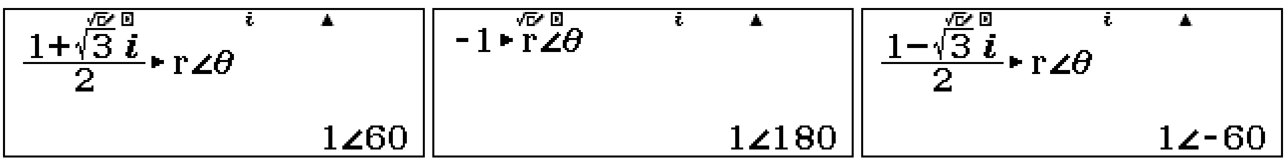
On an Argand diagram, square roots are opposite each other on a circle centred on the origin. This can be seen if the two roots are displayed in polar form. Check below that the two roots are both $2\sqrt{5}$ from the origin and that their arguments are 180^{o} apart.



You may have noticed in Module 4 that complex solutions are given to equations when necessary. For example, the three roots of $x^3 + 1 = 0$ are shown below in Equation mode:



As the equation can also be written as $x^3 = -1$, these are seen to be the three cube roots of -1. Back in Complex mode, when these solutions are expressed in polar form, it is clear that on an Argand diagram the three roots are equally spaced on a circle, 1 unit from the origin and 120^{o} apart from each other.



Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. A baby weighs 3.3 kg at birth. How much is that in pounds and ounces? (There are 16 ounces in a pound).
2. A website reported that it was 789 miles by road from Chicago to New York City. Use the calculator to convert this distance to kilometres.
3. Convert the decimal number 62 to hexadecimal, octal and binary notation.
4. Evaluate $12_8 + 57_8 + 14_8$.
5. Use binary logical operations to find 11010 and 10011. Explain your result.
6. Find both square roots of -9.
7. Evaluate $(6 - \boldsymbol{i})(4\boldsymbol{i} + 5)$.
8. Evaluate $(3 + \boldsymbol{i}) \div (2 + \boldsymbol{i})$.
9. Find the conjugate, the absolute value and the argument of $4 - 7\boldsymbol{i}$.
10. Express $12 + 5\boldsymbol{i}$ in polar form.
11. Express $\left[6, \frac{2\pi}{3}\right]$ in coordinate form.
12. Evaluate $(3 + 7\boldsymbol{i})^5$.
13. Evaluate $(4 - \boldsymbol{i})^{-3}$.
14. Use De Moivre's Theorem to find both square roots of $32 - 24\boldsymbol{i}$.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. Notice that the metric conversions on the calculators are in pairs. For example one conversion changes feet to metres, while a companion conversion changes metres to feet.

   Find the two conversion factors. How are the two conversion factors for each pair related to each other?

   Examine several pairs to reach your conclusions.

2. What is the effect of multiplying a number by its base? You have probably already noticed this for decimal numbers multiplied by ten, such as 23 x 10 = 230, but investigate it for other bases as well.

   For example, multiply 1011_2 by two, 234_8 by eight and $3A2D_{16}$ by sixteen.

   Try enough examples to see consistent patterns and then explain what you observe.

3. Compare octal and binary representations of numbers carefully. For example, the binary representation of the octal number 534 is 101011100; notice that if this binary number is split into three parts, each with three digits, 101, 011 and 100 that these are the binary representations of 5, 3 and 4 respectively.

   Try this idea with some other octal numbers.

   Then compare binary and hexadecimal numbers in similar ways.

4. How does the product of a complex number and its conjugate compare with the absolute value or the modulus of the number?

   Start by trying some examples such as $4 + 3\boldsymbol{i}$ or $6 – 2\boldsymbol{i}$.

   When the relationship is clear to you, explain why it occurs and how it is related to rationalising the division of one complex number by another.

5. Explain why multiplying a complex number by $\boldsymbol{i}$ has the effect of rotating its position on an Argand diagram 90^{o} anti-clockwise.

6. Find all three cube roots of -8, which are the solutions of the equation $x^3 + 8 = 0$. Use De Moivre’s Theorem to do this and then check the result by using the Equation mode of the calculator to solve the equation. (See Module 4 for details of this mode if necessary).

   Plot your solutions on an Argand diagram and observe their relationships to each other.

   Repeat this process for finding roots of some other numbers.

Notes for teachers

In this module, the use of the *ClassWiz* to handle measurement conversions and scientific constants, to represent numbers in different bases and to deal with complex numbers are all explored. Many students will need only some parts of the module, depending on their mathematics course. The text of the module is intended to be read by students and will help them to see how the calculator can be used for these various specialised purposes. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. 7 pounds 4 ounces. (Convert kilograms to pounds and then multiply pounds by 16 to change to ounces) 2. 1270 km (to the nearest km) 3. $3E_{16}$, 76_8 and 111110_2. 4. 105_8 5. 10010, which shows that the only digits that are 1s for each of the two numbers are the first and the fourth from the left. 6. 3***i*** and -3***i***. (The calculator gives only the first of these). 7. 34 + 19***i*** 8. $\frac{7}{5} - \frac{i}{5}$

9. 4 – 7***i***, √65, 1.052 10. [13,0.395] 11. -3 + 3√3***i*** 12. 23028 – 11228***i*** 13. $\frac{52}{4913} + \frac{47}{4913}i$

14. 6 – 2***i*** and -6 + 2***i***

Activities

1. This activity will allow students to see that the two conversion factors are reciprocals of each other. To find a factor, they will need to 'convert' a measure of 1, using the appropriate conversion. The two factors can be seen to be related to each other most easily by the use of the x^{-1} key. It is also possible to convert a measurement in one direction and then to immediately convert it in the other direction, to retrieve the original measurement. [Answer: conversion factors are reciprocals of each other.]

2. Students will recognise the procedure for multiplying decimal numbers by ten (i.e. adding a zero), but few students will have a good understanding of why this works. Note that, in HEX mode, 16 is represented by 10. [Answer: Multiplying by the base of a number system has the effect of 'adding a zero', regardless of the base, as each digit is 'moved' one place by the process.]

3. In this activity, students can see an interesting connection between numbers in different bases, which are powers of each other. They will find that the hexadecimal case involves blocks of four digits, while the octal case involves blocks of three digits. Encourage those who have completed the activity to speculate about numbers to base 4, which the calculator does not handle directly.

4. The product of a complex number and its conjugate has no imaginary part, which is why it is convenient for rationalising division (as shown on the seventh page of the Module itself). [Answer: Product of a number and its conjugate is the square of the absolute value of the number.]

5. For this activity, students should be advised to sketch some Argand diagrams and check the results of multiplying several complex numbers by ***i***. This will help them to see intuitively that multiplying by ***i*** has the effect of changing the real parts of a number to complex parts and changing the imaginary parts to be real parts. In effect this switches the positive *x*-axis with the positive *y* axis and the positive *y* axis with the negative *x*-axis, or rotating 90 degrees anticlockwise.

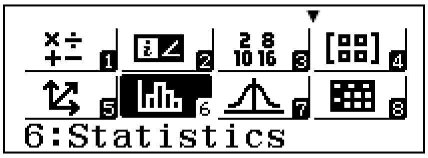
6. Check that students know how to use Equation mode, explained in Module 4. [Answer: The three cube roots of -8 are -2, 1 – √3***i*** and 1 – √3***i***. The three roots are equally spaced around an Argand diagram and all have the same absolute value (of 2) and thus lie on a circle. This result holds for other numbers and a similar result holds for other roots.]

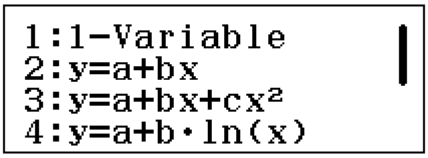
Module 10
Univariate statistics

Statistical data analysis requires technological support for efficiency and you will find that the *ClassWiz* supports univariate statistics well. In this and the next module, we will use Statistics mode, focusing on univariate statistics in this module and bivariate statistics in Module 11.

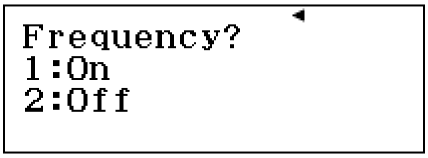
Getting started with statistics

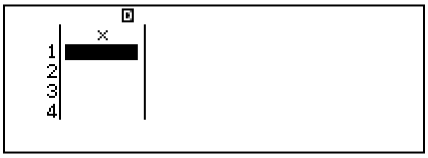
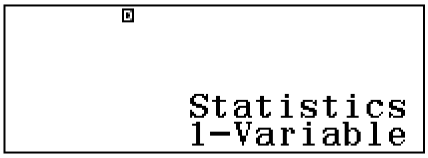
Univariate statistics involves data on a single variable. Start by entering Statistics mode with MENU 6 and select univariate statistics with 1 (1-VAR). The other choices all involve bivariate data (with two variables) and will be used in the next module.





Univariate data sometimes have associated frequencies, so that each data value might be repeated several times. This will be discussed later in this module, but for now, we will assume that frequencies are not involved. To turn the frequencies information off, use SET UP, select the second page with ▼ and then tap 3 Statistics.



Select 2 Off and you will then see a blank data table for entering the data as shown in the screen above. Notice that tapping the AC key will advise you that the calculator is set to deal with univariate statistics.

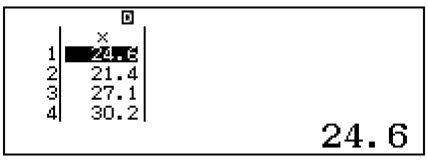
Entering, editing and checking data

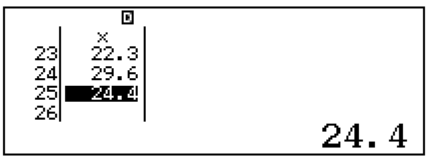
The calculator will allow up to 160 data points for a single variable to be entered, or up to 80 points if each point has a frequency entered. If you have more than 160 items of data, you will need to use frequencies (as described later in this module.)

To illustrate the use of the calculator for univariate analysis, consider the data below, which were obtained from an experiment on growing beans from seeds. The heights of a number of bean plants were measured (in centimetres), six weeks after they were planted, with the results shown below:

24.6 21.4 27.1 30.2 20.4 20.7 21.8 29.1 22.5 21.6 31.8 21.0 17.1 27.7 28.1 24.9
24.7 24.0 23.6 19.1 20.8 24.8 22.3 29.6 24.4

If the data table is not on the screen, start by tapping OPTN 3 Data. Enter these measurements into the calculator in the *x* column, tapping the = key after each one. Notice that, after you tap =, the cursor moves down to the next row of the table.



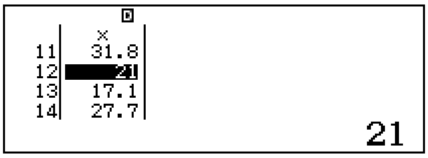


Typing errors are always possible, so that it is wise to check the entries.

If you make an error in entering a value before tapping the [=] key, you can correct it using the [DEL] key, enter the correct value and then tap [=]. If you notice an error in an entered data point, use the cursor to highlight the incorrect point and retype it with the correct value.

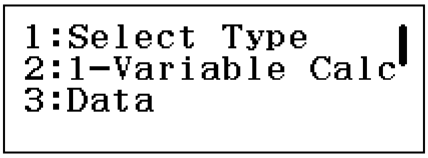
As the data are entered, you can scroll up and down using [▲] and [▼] to do this. You can scroll in either direction and, in particular, can scroll down from the bottom value to the top value, or scroll up from the top value to the bottom value (as if the data were in a loop). As you scroll, you will see that highlighted values are shown in greater detail and size at the bottom of the screen than they are in the table, as with Table mode in the calculator.

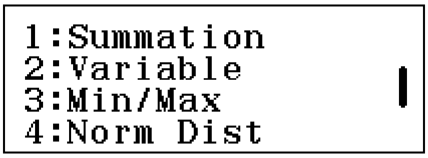
Notice also that the calculator has entered 21 instead of 21.0 and 24 instead of 24.0 respectively, even though the values were typed with a decimal point.



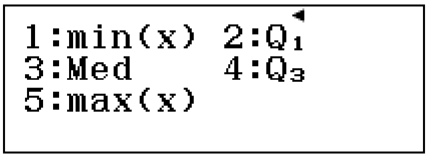
Once all data are entered, an easy check involves the number of entries. In this case, the final data point is marked as the 25th point, which matches the number of entries in the list above.

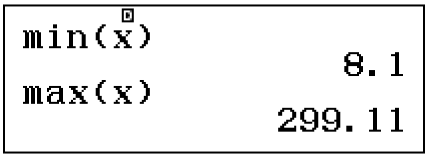
If the correct number of data points has been entered, tap [AC] to leave the data editing and then [OPTN] to see the Statistics options, which are shown in the two screens below.



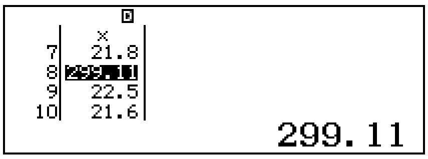


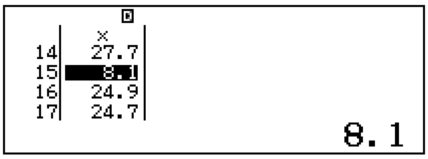
It is a good idea to check the maximum and minimum entries, since these can sometimes represent typing errors. To access these from the OPTN menu shown above, tap [3] *MinMax* and select either the minimum or maximum values. In this case, each of these is shown below:



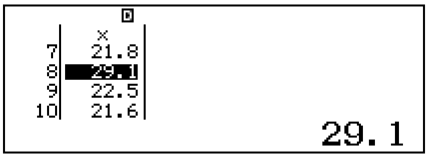


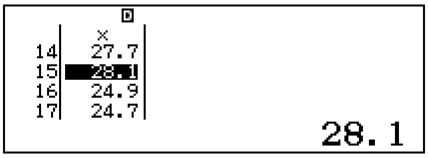
We have entered each of these two values incorrectly, as you can tell from a quick look at the original data, where all of the values are 2-digit numbers with a single place of decimals. In this case, typing errors have been made (although you are unlikely to have made the same errors in your calculator).





To correct any errors, tap [OPTN] [3] to return to the Data list. In our case, a quick scroll indicates that the 8th measurement of 29.1 has been entered incorrectly, and the 15th measurement of 28.1 has also been entered incorrectly. Each of these can now be corrected by entering the correct value in place of the incorrect one:





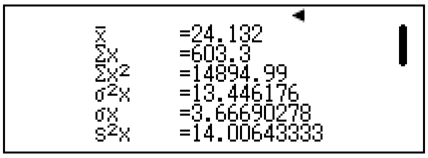
It is always prudent to check data accuracy before undertaking any statistical analysis.

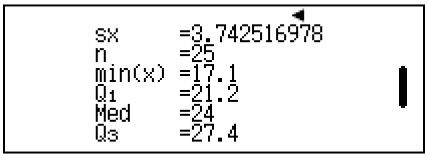
When the data list is showing, tap OPTN 2 *Editor* to add a data point if necessary.

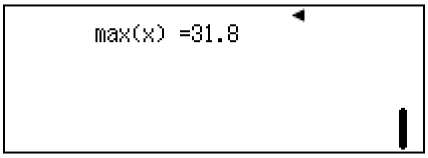
Once the data are entered, you can edit them using the OPTN 3 command. *Be careful*, however, that you do not leave Statistics mode (even if you re-enter it immediately with MENU 6) or your data will be deleted, as the calculator expects a fresh set of data.

Retrieving statistics

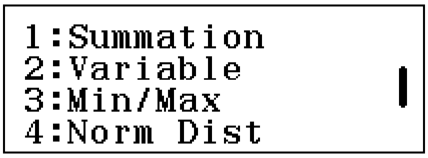
Once you are sure that data have been entered correctly, appropriate statistics can be obtained via the OPTN menu. Tap AC to leave the data table tap and OPTN 2 to display all the 1-variable calculations; three successive screens are needed to show all these, as shown below. (Notice the cursor symbol at the top, indicating that you can return to the previous menu with ◀ if you wish.)

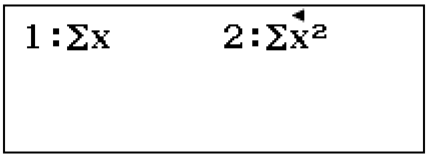


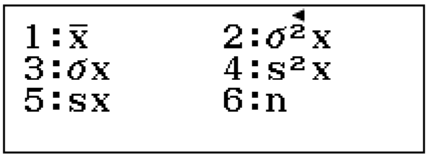




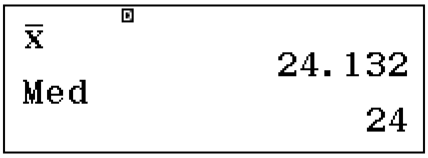
As you have already seen with the *Min/Max* menu, these statistics can also be obtained individually through the second screen of the OPTN menu. The *Min/Max* menu was used to check the data earlier, while the screens below show the *Summation* and *Variable* menus.





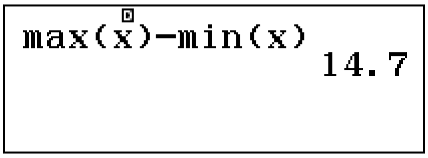


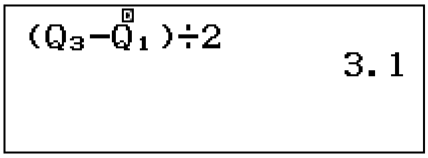
For example, the *mean* of the data is represented by $\bar{x} = \frac{\sum X}{n}$ and is available in the *Variable* menu by tapping 1 and then =. In this case, the mean of 24.132 cm represents an average of the heights of the beans. The *median* of the data is the middle value, which can be obtained individually from the *Min/Max* menu below. In this case, the two measures are close to each other.



While the mean and median describe the location of the data, the spread of the data is also important. One measure of spread is the *range*, which is the difference between the minimum and the maximum values. Now that the data have been checked, the correct values of 17.1 and 31.8 for these are shown below, and also in the display of all statistics earlier.

In this case, you could calculate mentally the range of 14.7 from these, although it is also possible to do this on the calculator directly, using the individual values from the *Min/Max* menu, as shown below. Similarly, the *semi-interquartile range* of 3.1 can also be obtained from the upper and lower quartiles, using the individual values:

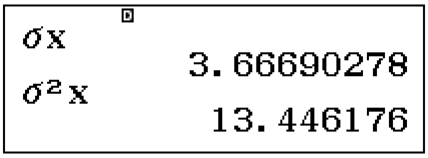




The range is a fairly crude measure of the spread of the data, since it uses only two values. The *standard deviation* of a *population*, represented by σ_x, also measures the spread of the data:

$$\sigma_x = \sqrt{\frac{\sum X^2 - \frac{\left(\sum X\right)^2}{n}}{n}}$$

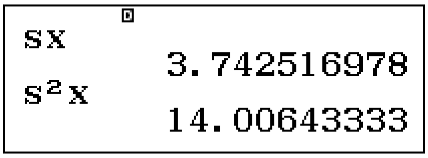
The standard deviation is based on all of the data, not just the two extreme values. In this case, if we had the whole population of beans, it would be approximately 3.67 cm, as shown below. The population *variance* of approximately 13.45 could be obtained by tapping the $\boxed{x^2}$ key after obtaining the standard deviation, or could be obtained directly from the Variable menu:



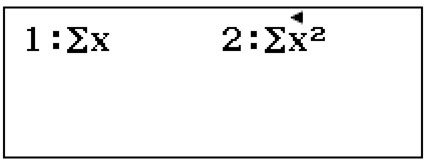
We do not often have access to the whole population, however, but instead have only a random sample from the population. In that case, the most appropriate measure of the standard deviation is the *unbiased estimate* of the population standard deviation, represented by s_x:

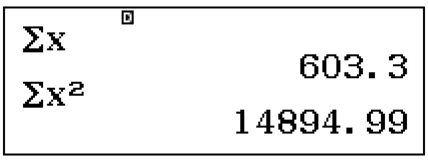
$$s_x = \sqrt{\frac{\sum X^2 - \frac{\left(\sum X\right)^2}{n}}{n-1}}$$

In general, as you see from studying the two formulas, s_x is a little larger than σ_x , because the numerator is divided by a smaller denominator. The *sample standard deviation* s_x is appropriate to use if we regard the beans as a sample from a population. In this case, the sample of 25 beans has a sample standard deviation of 3.74 and a sample variance of 14.01.



As well as these statistics, sometimes it is helpful to obtain the sums of the original scores and the sums of their squares, as these are the values used internally by the calculator to undertake the calculations for the standard deviation. These are shown below from the *Summation* menu:





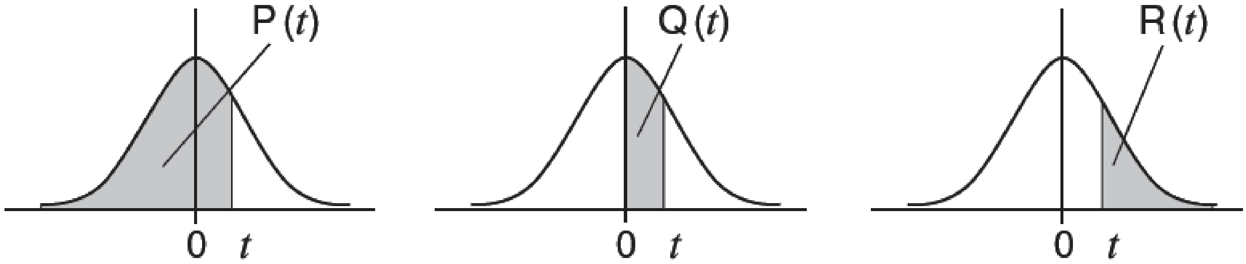
The statistics provided by the calculator for these data suggest that the bean heights are spread approximately symmetrically with most heights within about two standard deviations of the mean. (A graph of the data would also help to interpret the distribution, of course.)

Determining normal probabilities

(Module 12 describes the normal probability distribution in some detail. You may wish to skip this section until you have completed Module 12.)

The calculator will allow you to use the normal probability distribution with the data that have been entered for the beans, assuming that we have the whole population of beans. The distribution of heights has a mean of $\mu = 24.132$ and a standard deviation of $\sigma_x \approx 3.667$.

To use the normal probability distribution for the bean heights, assuming that they follow a normal distribution, requires a transformation to a distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$. Once a bean height is transformed, probabilities $P(t)$, $Q(t)$ and $R(t)$ associated with the height can be determined from the calculator, as shown on the next page.



To transform a bean height *X* to the standard normal variable, represented in the calculator and in the diagrams above by *t*, the following transformation is used by the calculator:

$$t = \frac{X - \mu}{\sigma_x}$$

In this particular case, the transformation used is $t = \dfrac{X - 24.132}{3.66690278}$

It is common in textbooks for the symbol *z* to be used instead of *t*.

Fortunately, the calculator can perform this transformation automatically via the normal distribution commands, obtained by tapping [4] *Norm Dist* in the OPTN menu:

```
1:Summation
2:Variable
3:Min/Max
4:Norm Dist
```

```
1:P(      2:Q(
3:R(      4:▸t
```

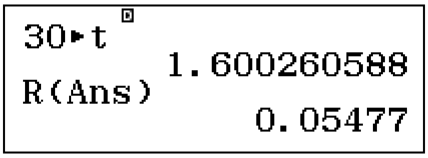
The appropriate transformation is carried out by first entering a value for *X* and then tapping [4] in the normal distribution menu. For example, the screen below shows how to find the *t*-value associated with a bean height of $X = 26$:

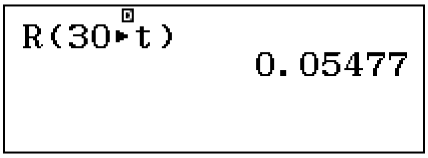
```
26▸t
        0.5094217414
P(Ans)
             0.69477
```

To find what proportion of heights is expected to be up to 26 cm, we would need to evaluate P(0.5094217414). However, it is much easier to use the calculator [Ans] capability in this case, since the *t*-value is already shown on the screen. The screen above shows the command needed to show that about 69% of bean heights from this population are expected to be less than or equal to 26 cm.

(In fact, 18 of the 25 heights – 72% of this particular sample – were in fact less than or equal to 26 cm.)

Other questions can be answered similarly. For example, to find the probability that a randomly selected bean height will be greater than 30 cm, first evaluate the associated *t*-value, then use this to find the appropriate probability, *R*(*t*):





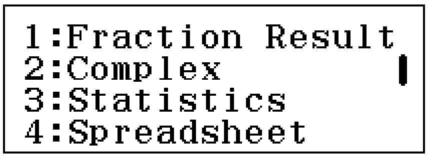
The second screen above shows that it is possible to determine probabilities like this with a single command, if you wish, although this is not a good idea if you would like a record of the *t*-value. In this case, the probability obtained suggests that only around 5% of beans are expected to be higher than 30 cm. (In fact, two of the 25 heights – 8% of this particular sample – exceeded 30 cm.)

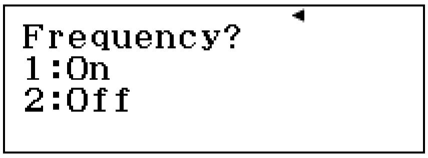
Frequency data

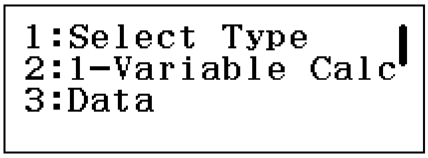
Sometimes, a univariate data set will comprise more than 160 observations, the limit for the calculator, so that grouped data with frequencies will be needed. Some univariate data are routinely collected with frequencies, which are also best handled in the calculator using frequencies.

When frequencies are used, there is a limit of 80 data points (each with an associated frequency). In effect, this means that your calculator can handle larger numerical data sets, provided only that they can be organised into no more than 80 separate values.

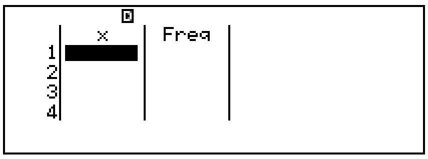
To activate frequencies, use the second page of the SET UP menu, as shown below and select [3] to access the Statistics Set Up and then [1] to turn the frequencies on. Then use the [OPTN] key to select 1-variable statistics again, necessary after changing the number of data columns. *Notice that this change will erase any data already in the data table.*







When you access the data using the Statistics menu, you will see that the data are cleared and that there are now two columns, one for the data and a second for the associated frequencies:



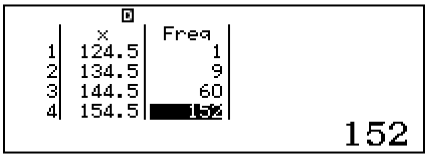
To see how the calculator handles frequency data, consider the following table which shows data regarding the heights of a group of 455 girls in a school. Each girl's height was reported by the girls themselves to the nearest centimetre. As there are more than 160 girls and more than 80 different heights reported, grouping of the data was necessary to enable analysis. Data were then recorded in the intervals shown.

Other grouping methods are of course possible, but usually six to ten intervals are used.

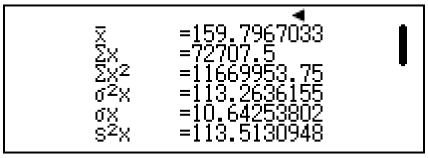
To enter the data into the calculator, each interval is represented by its midpoint as a separate value of the height variable, *X*. After a value is entered, and the = key tapped, the cursor moves down the column involved. For this reason, it is easier to enter the data in columns (all the midpoints first and then all the frequencies), but make sure that the *X* values and the frequencies match up.

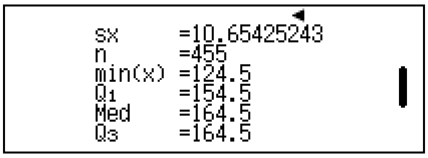
Height (cm)	Midpoint	Frequency
120 - 129	124.5	1
130 - 139	134.5	9
140 - 149	144.5	60
150 - 159	154.5	152
160 - 169	164.5	161
170 - 179	174.5	57
180 - 189	184.5	13
190 - 199	194.5	2

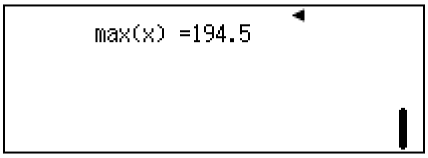
A good check on data entry is that the frequencies and the data are correctly aligned:



Once all the data are entered, tap AC. Retrieving statistics is done the same way via the OPTN key as for data without frequencies. The key values are shown below.

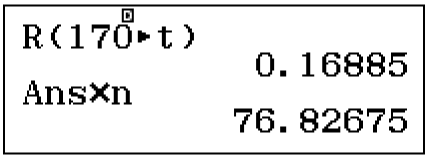






In this case, notice that the total number of data points is n = 455, although only eight values were entered, which verifies that the frequencies have been entered correctly.

The mean and standard deviation can also be used automatically by the calculator to examine normal probability distributions. For example, the screens below suggest that, assuming that the heights follow a normal distribution, about 17% or 77 of the girls would be taller than 170 cm.

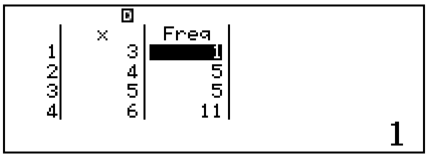


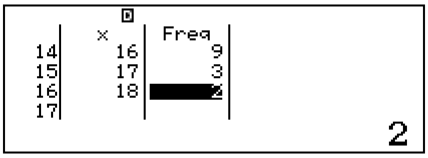
In fact, the data table shows that only 72 girls in the data set were taller than 170 cm, a few less than expected. However, caution is needed to expect too much precision in situations like this, as the data will only be likely to be a*pproximately* normal.

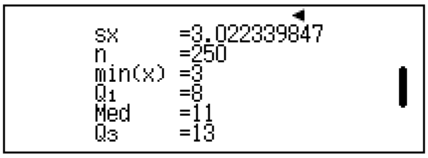
Frequencies are not only relevant when data have been grouped. Sometimes, they are used because the data are naturally collected with frequencies. For example, to examine the likely effects of tossing three standard 6-sided dice and adding their totals, a class combined efforts and generated the following distribution of results of 250 sets of tosses, with each of the 25 students rolling three dice ten times.

Total	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Frequency	1	5	5	11	15	29	21	22	39	31	28	16	13	9	3	2

These data can be entered into the calculator, starting with MENU 6 and 1, which will clear the previous data. The frequency setting will remain unchanged, and thus allow you to enter the data and their associated frequencies.

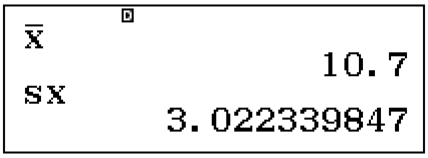






It's a good idea to check the data entry. One way to do so is to retrieve the number of data points from the 1-Variable Calculation menu, as shown above. In this case, n = 250 checks.

Again, the main interest here is in the mean and standard deviation, which are both shown below:



Clearly, it is a great deal easier to summarise these data numerically using the calculator than it would be with hand calculations. The result for the mean is not unexpected, as the mean result for tossing a single die is 3.5, so we might expect that the mean result of tossing three dice independently would be about three times 3.5, or around 10.5.

Inferential statistics

In practice, relatively small data sets are not unusual in statistics, since obtaining data is frequently expensive so that people will try to use data from samples to predict the population from which the sample is drawn. Making inferences from a random sample to a population always relies on assumptions about the data and the sampling process.

A very important result, too advanced to describe here in detail, is that when samples of size n are drawn at random from a population with mean μ and standard deviation σ, the means of the samples are likely to follow approximately a normal distribution with the same mean μ as the population and with a standard deviation, usually called the *standard error of the mean*, or often just the *standard error*, that is defined as:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \ .$$

This result can then be used to give a good estimate of the likely interval within which the population mean lies, using some properties of the normal probability distribution.

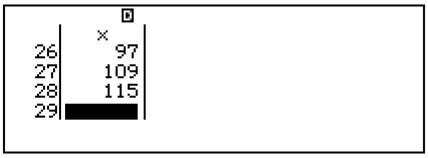
For example, roughly 95% of the time, it is known from the normal distribution that the population mean will be within 1.96 standard errors of the sample mean.

In the same way, roughly 90% of the time, it is known from the normal distribution that the population mean will be within 1.645 standard errors of the mean.

Here is an illustration of this idea. Suppose a scientist wishes to determine the typical mass of a new species of fish in a river in a remote location. She obtained a random sample of 28 fish by trapping them in the river and weighed them carefully. Here are the masses, given in grams:

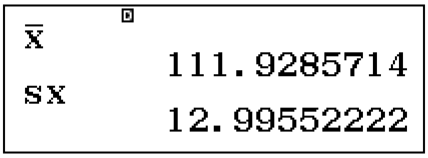
112	94	141	102	112	91	125	114	106	121	130	113	99	119
107	127	94	100	104	116	137	105	105	111	128	97	109	115

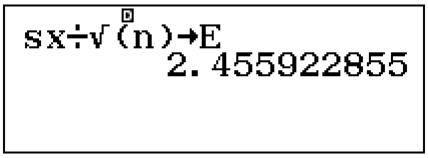
Enter these data into the calculator, after first turning off the frequencies in the SET UP menu.



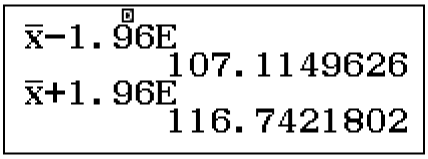
The screens above provide a quick check on the accuracy of data entry: there are 28 masses entered, the minimum is 91 g and the maximum is 141 g. These seem to be consistent with the data.

The mean and standard deviation are obtained from the OPTN menu, bearing in mind that the sample standard deviation is an estimate of the (unknown) population standard deviation, and so is used to get a good estimate of the standard error:



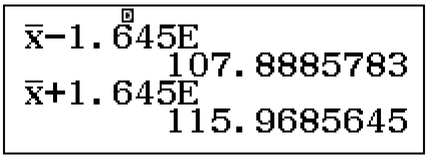


The estimate of the standard error can be obtained using values available in the *Variable* menu, as shown above. For convenience, this result has been stored into memory E, using the command [STO] [COS], although you might choose to just write it down on paper. Then the confidence interval is given by the two endpoints shown below, representing the sample mean ± 1.96 x standard error:



So the scientist can be 95% confident that the mean mass of the population from which her fish were sampled is between about about 107 and 117 grams.

A smaller confidence interval will mean that she is a little less confident of her prediction for the mean. So, she can be 90% confident that the mean mass of the population from which her fish were sampled is between about 108 and 116 grams:



In other words, about nine times out of ten, these procedures will generate a confidence interval that includes the *actual* population mean of the mass of the new species of fish. (So, about one time in ten, the actual mean will lie *outside* the 90% confidence interval that is obtained.)

If a larger sample were to be taken, then the standard error would be smaller and so the confidence interval would also be smaller than would be the case for a smaller sample. For this reason, scientists and statisticians will generally prefer to take a larger sample if possible, so that a confidence interval is smaller, giving more precise information about the likely population mean.

Ideas in inferential statistics are quite sophisticated and you should not rely on this brief treatment to understand them thoroughly, but are advised to study them elsewhere as well.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. A farmer recorded the money received on each day of the week at a Farmers' Market stall:
   $102.50, $250.00, $310.20, $150.70, $207.40, $120.90, $210.00

   Use the calculator to find the mean and standard deviation of the daily takings for that week.

2. A receptionist recorded the number of phone calls received each day of a week, as follows:

Day	Mon	Tue	Wed	Thu	Fri
Number	140	125	134	132	1260

   (a) Find the mean and the standard deviation of the daily number of calls for that week.

   (b) After completing her calculations, the secretary realised that Friday's data had been wrongly recorded: the correct figure was 126, not 1260. Edit the data to correct this error and then find the mean and standard deviation of the corrected data.

3. A random survey of students in a local high school from Year 7 to 12 was conducted to find out how much money students brought to school with them. The following data (in $) were obtained:

   5.20, 6.15, 0.40, 10.55, 2.56, 5.12, 16.40, 25.30, 16.20, 1.45, 6.35, 10.10, 15.20, 18.75, 2.30, 0.80, 1.20, 6.90, 8.50, 2.30

   (a) Enter these data into the calculator and retrieve all available statistics from the OPTN menu; find the mean, the median, (unbiased) standard deviation, the lower quartile and the range.

   (b) Assuming the data are normally distributed, use the calculator to find the *t*-value associated with a money value of $6.00.

   (c) Assuming the data are normally distributed, use the calculator to find the probability that a randomly chosen student brings less than $6 to school with them.

4. If we can assume that the distribution of amounts of money carried by students to school is normally distributed with $\mu = 8$ and $\sigma = 7$, how many students out of 20 would you expect to carry less than $3.00?

5. The numbers of pets owned by 70 grade 10 students are shown below:

No. of pets	0	1	2	3	4	5	6	7	8
No. of students	12	25	12	10	6	2	1	0	2

   Use frequencies in the calculator to find the mean number of pets per student.

6. An artist painted three paintings every month for a year.

   (a) Without using frequencies, use the calculator to find the mean and standard deviation of the number of paintings per month.

   (b) Repeat the calculations of part (a), this time using frequencies.

   (c) Without using the calculator, determine Σx and Σx^2 for these data. Then check that the calculator result is the same as yours.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. Two groups of Grade 8 students were given the same test at the same time. Here are the scores of the two groups:

   Group A: 5, 5, 5, 7, 4, 5, 5, 7, 8, 8, 7, 6
   Group B: 0, 9, 1, 4, 3, 10, 10, 8, 3, 4 10, 10

   (a) Find the means and standard deviations (s_x) of these two groups. Comment on any similarities and differences observed.

   (b) Find another set of scores with a mean score of 6 and different score values from these.

2. Two classes of 30 students conducted an experiment that involved rolling a die and recording the results as a class. Each student rolled a single die ten times, with the following results:

Number on die	1	2	3	4	5	6
Frequency	47	55	45	54	55	44

   A second class of 30 students conducted the same experiment with these results:

Number on die	1	2	3	4	5	6
Frequency	54	45	48	56	48	49

   (a) Find the mean and standard deviation (s_x) of the results of each class.
   (b) In pairs or groups, roll a die a total of 300 times and record the results. Compare the mean and standard deviation with those of the two classes.
   (c) Investigate the effects on statistics of combining the two class results into a single group with total frequency of 600.

3. An Australian retailer of household equipment employs staff with different annual salaries: General manager ($100 000), two Sales staff ($60 000), a personal assistant ($50 000), two clerical staff ($40 000), a warehouse organiser ($40 000), a delivery driver ($35 000) and a receptionist ($30 000). Find the mean, median (i.e., middle) and mode (i.e., most frequent) of these annual salaries. If the General Manager awards himself a pay rise of $100 000 to get a new salary of $200 000, calculate these statistics again. What do you notice?

4. The weights of a species of pygmy possums are normally distributed with $\mu = 45$ g and $\sigma = 1.8$ g.
   (a) If you picked a pygmy possum at random, what is the probability that it would weigh less than 43 g?
   (b) Use your calculator to estimate the weight below which 5% of the pygmy possums lie.

5. Use your calculator to explore the effects of transforming values. Enter a small data set such as {2, 3, 5, 7, 8} into the calculator and record the mean and standard deviation on paper.
   (a) Add 3 to each data point and find the mean and standard deviation again. Compare with the original statistics.
   (b) Multiply each data point by 4 and find the mean and standard deviation again. Compare with the original statistics.

6. Experiment with your calculator to construct a data set with ten elements for which the mean is 50 and the standard deviation is 10.

Notes for teachers

This module highlights how *ClassWiz* in Statistics mode can support students to think about univariate data analysis. The text of the module is intended to be read by students and will help them to see how the calculator can be used to examine data in various ways. Note that the material on inferential statistics might be omitted for younger students. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. $\bar{x} = 193.1$, $\sigma = 68.31$, $s = 73.78$ (a) 358.2, 504.15 (b) Use SHIFT 1 2 to edit data. 131.4, 6.15
3. (a) $\bar{x} = 8.09$, $Mdn = 6.25$, $s = 7.00$, $Q_1 = 2.3$, range = 25.3 – 0.4 = 24.9 (b) Use the Norm Dist menu to get $t = -0.31$ for $x = 6$ (c) and then and then P(*Ans*) = 0.38 4. 23.75% or about 5 5. 1.93
6. (a) 3, 0 (b) the same result: 3, 0 (c) $\Sigma x = 12 \times 3 = 36$, $\Sigma x^2 = 12 \times 9 = 108$.

Activities

1. This activity illustrates how standard deviations describe variability, since the two groups have the same mean of 6, but the first is less variable than the second. To make a new set with mean 6, encourage students to edit an existing set using the Statistics menu with SHIFT 1 2 and notice that the mean is unaffected provided the total Σx stays the same. [Answers: $\bar{x} = 6$, $s_A = 1.35$, $s_B = 3.86$.]

2. This activity will help to show that random data are not always the same, and it is helpful to gather real data as well as use secondary data. With sufficient data, results will be quite consistent, although not identical. It is instructive that combining the group data does not affect the mean or the variability of the dice scores. [Answers: (a) 3.49, 1.68 and 3.49, 1.71 (c) combined 3.49, 1.69]

3. The main point of this activity is to examining the effects of outliers on statistics. It is assumed that students already know how to find by hand the mode and the median (neither of which is addressed by the calculator), but will see the dramatic effects of outliers on the mean (only) by experimenting with other possible salaries for the General Manager, editing the data set. You may like to replace Australian salaries with relevant local salaries. [Answers: median and mode stay at \$40 000 but mean rises from \$50 556 to \$61 667.]

4. This activity involves using the normal probability tables with summary statistics rather than data. It is not necessary for data to be entered to use the commands, as discussed in Module 12. For the second part of the activity, encourage students to use trial and adjustment to get an approximate answer; encourage students to use the replay facility with ◀ to do this efficiently. For greater efficiency, the calculator has an inverse normal probability capability, explored in the Probability module. [Answers: (a) $P((43 - 45) \div 1.8) \approx 0.13$ (b) about 42.0 g]

5. Students can explore transformations of data directly to see their effects on statistics. Encourage students to experiment with other operations to understand their effects, including using pairs of operations, such as subtracting 2 and then multiplying by 5. This will help them to see the significance of the important normal transformation of $z = (X - \mu)/\sigma$, which is represented in the calculator as t. [Answers: originally $\bar{x} = 5$, $s \approx 2.55$ (a) $\bar{x} = 5 + 3 = 8$, s is unchanged (b) $\bar{x} = 5 \times 4 = 20$, $s \approx 10.20$ (s is multiplied by 4).]

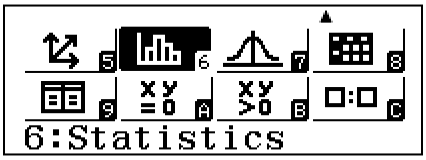
6. There are many possible data sets that meet the requirements, at least approximately, and this activity is intended for students to experiment to learn intuitively from experience how the data values affect the statistics. A symmetrical set of ten values around 50 will result in a mean of 50, while spacing out the values will produce various standard deviations. [Answer: One possible example is the set {35,39,42,45,48,52,55,58,61,65} with $\bar{x}$=50 and $s \approx 9.9$.]

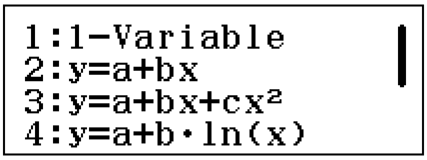
Module 11
Bivariate statistics

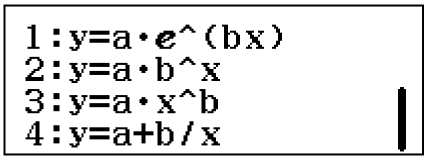
Statistical data analysis requires technological support for efficiency and you will find that the *ClassWiz* supports bivariate statistics well. As for the previous module, we will use Statistics mode, focusing on bivariate statistics. Many of the calculator operations are similar to those used for univariate statistics.

Getting started with bivariate statistics

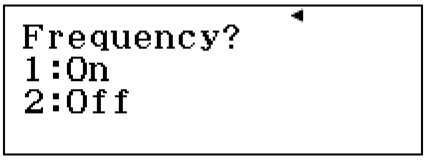
Bivariate statistics involves data with two variables, so that interest is generally on the relationship between the two variables. The calculator assumes that the variables are named x and y respectively. Start by entering Statistics mode with MENU 6. Each of the choices (except the first one) involves bivariate statistics. The choices refer to seven different models for representing the relationship between the variables. To begin with select 2, which allows us to explore a *linear* relationship in the form of $y = a + bx$.

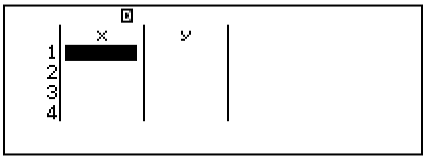






Although it is rare, bivariate data sometimes have associated frequencies, so that each data pair might be repeated several times. We will assume for now that frequencies are not involved. To turn the frequencies information off, use SET UP, select the second page with ▼ and then tap 3 *Statistics*.





Select 2 *Off* and you will then see a blank data table for x and y as shown in the screen above.

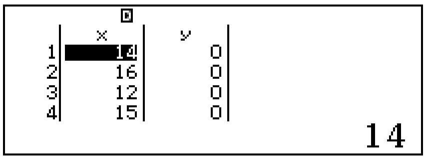
Entering, editing and checking data

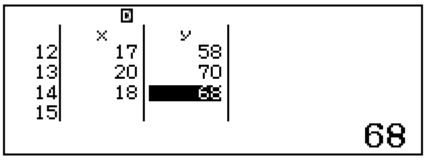
The calculator will allow up to 80 data points for each of two variables to be entered. If you have more than 80 pairs of data, you will need to use frequencies.

To illustrate the use of the calculator for bivariate analysis, consider the data below. Nurses in a school were checking children's pulses and wanted to know whether good readings could be obtained after only 15 seconds, as they expected this would save them a lot of time. So they measured the number of heartbeats of a group of 14 children for 15 seconds and then measured the number of heartbeats again for 60 seconds. They obtained the following results:

x (15 secs pulse)	14	16	12	15	13	19	14	25	22	23	24	17	20	18
y (60 secs pulse)	57	65	43	59	41	75	51	92	84	87	86	58	70	68

It is efficient to enter these measurements into the calculator in the x-column first, tapping the = key after each one. Notice that, after you tap =, the cursor moves down to the next row of the table, and stores a 0 in the y column, as shown below. Then enter the y-values in the same way, after first moving the cursor using ▶ and ▲.





Typing errors are always possible, especially if a large number of data points are entered, so that it is wise to check the entries.

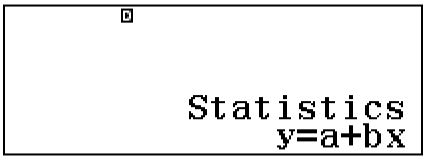
If you make an error in entering a value before tapping the [=] key, you can correct it immediately using the [DEL] key, enter the correct value and then tap [=]. If you notice an error in an entered data point, use the cursor to highlight the incorrect point and retype it with the correct value.

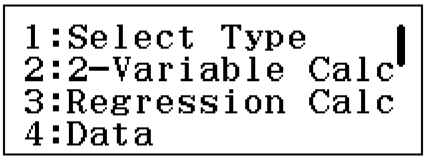
As the data are entered, you can scroll up and down using ▲ and ▼ or left and right using ◀ and ▶. You can scroll in either direction and, in particular, can scroll down from the bottom value to the top value, or scroll up from the top value to the bottom value (as if the data were in a loop). As you scroll, you will see that highlighted values are shown in greater detail and size at the bottom of the screen than they are in the table, as with Table mode in the calculator.

An easy check on data entry involves the number of entries. In this case, the final data pair of (18,68) is marked as the 14th point, which matches the number of entries in the data table above.

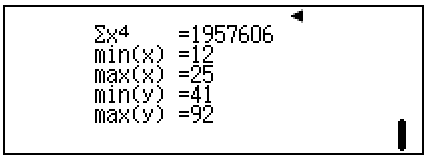
If you wish to enter a new data pair, while the data table is showing, tap [OPTN] [2] *Editor*.

Once all data are entered, tap [AC] and [OPTN] to enter the Statistics menu, the first page of which is shown below.





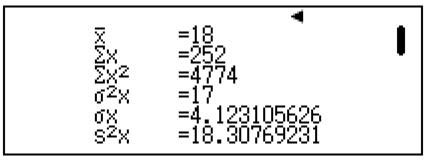
It is a good idea to check the maximum and minimum entries, since these are often of interest, allow you to calculate the range but can sometimes represent typing errors. To access these from the OPTN menu, above, tap [2] *2-Variable Calc* and use ▼ to go down to the fourth page. All four minimum and maximum values for *x* and *y* are shown below.

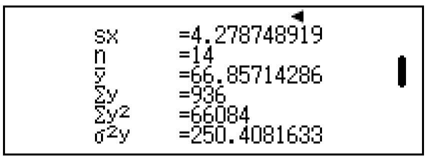


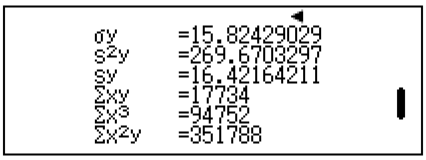
Each of these four values is correct in this case. To correct any errors, tap [AC] and then [OPTN] [4] *Data* to return to the data table for suitable editing.

Retrieving statistics

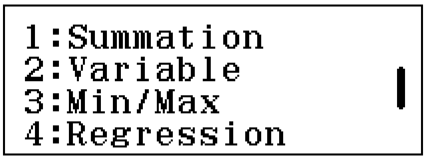
It is always important to check data accuracy before undertaking any statistical analysis. Once you are sure that data have been entered correctly, appropriate statistics can be obtained via the OPTN menu. Tap [AC] to leave the data table and [OPTN] to display the menu. Four pages of statistics are then available with [2] *2-Variable Calc* (the fourth page is shown above)

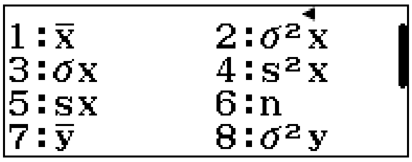


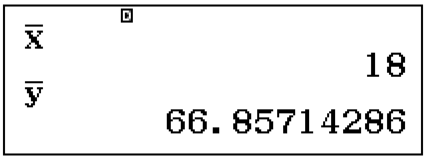




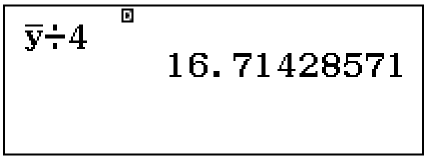
The second page of the OPTN menu allows you to examine and use individual statistics. The mean of the shorter (15-second) pulses, represented by $\bar{x}$, is available by tapping $\boxed{2}$ *Variable* and then $\boxed{1}$ $\boxed{=}$. The mean of the longer (60-second) pulses can be obtained by first returning to the *Variable* menu in the second page of the OPTN menu and then tapping $\boxed{7}$ $\boxed{=}$:







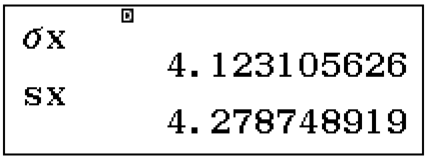
You can still use the calculator while in Statistics mode. For example, to calculate one quarter of the mean of the longer pulses immediately after obtaining the result, use the replay key with ◀ and edit the expression, as shown below:



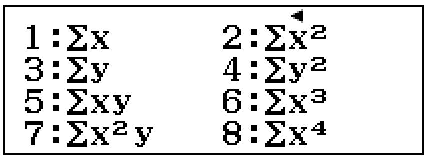
Although this is close to the mean of the 15-second pulses, it is a little smaller than that mean, perhaps suggesting that the shorter readings are a little higher than might be expected.

As for univariate statistics, there are two measures of standard deviation available for each variable, with σ measuring the population standard deviation and *s* providing an estimate for a sample, as explained in Module 10.

In general, *s* is a little larger than σ:



As well as these statistics, sometimes it is helpful to obtain the sums of the original scores and the sums of their squares, as these are the values used internally by the calculator to undertake the calculations for the standard deviation. These are shown below, obtained by first tapping $\boxed{3}$ *Summation*:

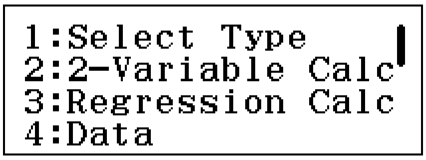


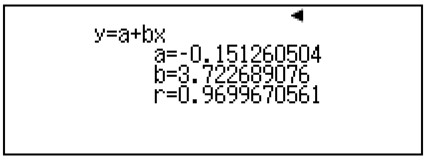
Relationships between some of these statistics and the calculation of variances and means were described briefly in Module 10; these various sums are used internally in the calculator for calculations. However, most people are generally comfortable with allowing the calculator to complete the computations, and do not make use of these statistics directly.

Using a linear model

The major reason for studying two variables at once is to understand the relationship between them. There are various kinds of relationships that the calculator allows you to explore. The most important of these involves a linear model of the form $y = a + bx$. (This is often written in schools in the form y = gradient x x + intercept; i.e. $y = bx + \mathrm{a}$.) At the start of this module, you chose this model in the opening screen for Statistics mode. The calculator provides the best-fitting model of this kind for the data entered, by providing values for a and b.

To access these, tap OPTN to display the OPTN menu for Statistics and then 3 to display the Regression calculations:



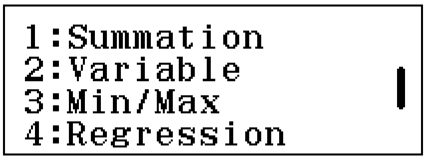


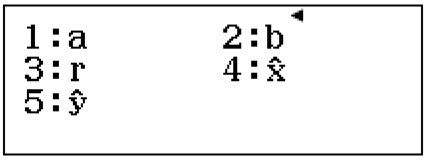
So, the best-fitting linear model for these data (rounded to two decimal places) is

$$y = -0.15 + 3.72x \quad \text{or} \quad y = 3.72x - 0.15$$

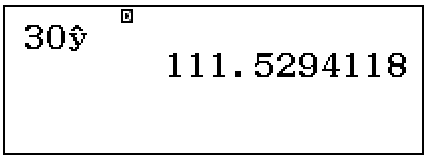
This is close to, but not quite the same as might be expected for the pulses that $y = 4x$, based on an assumption that the number of beats in 60 seconds would be four times the number in 15 seconds.

The next page of the OPTN menu allows you to access individual regression statistics, through 4 *Regression*.

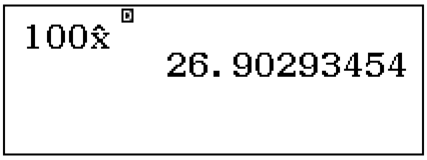




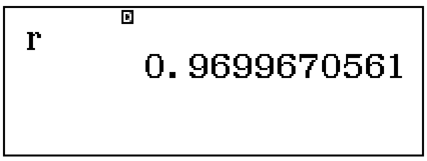
The calculator allows you to use the linear model automatically to predict y values for particular x values. For example, to predict the number of beats (y) in 60 seconds when the number of beats in 15 seconds is $x = 30$, enter 30 and then the $\hat{y}$ command in the Regression menu, followed by =:



(The caret mark over the y refers to an estimated value.) It is also possible to automatically use the linear model to predict an x value associated with a particular y value. For example, for $y = 100$, the $\hat{x}$ command predicts that $x = 26.9$:

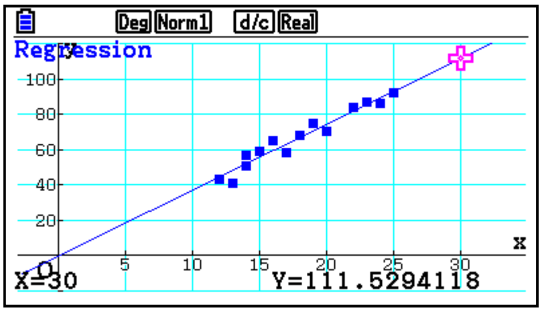


The calculator also provides a measure of how closely aligned the data are to the model studied. The statistic used is the *correlation coefficient*, represented by the symbol r, accessed in the Regression menu. The value always lies between -1 and 1, each of which represents a perfect fit to the model. In this case, the linear model is a good fit to the data, since r is very close to 1:



It is always a good idea to examine bivariate data visually, using a scatter plot, in order to see what is the apparent relationship between the variables. You can do this with graph paper, plotting all the points and the linear model.

A graphics calculator such as a CASIO fx-CG20 is also an efficient and helpful device to do this. The following screen shows the scatterplot and the linear model together, making it clear that the points are clustered close to the line and confirming that a linear model is a good choice in this case.



The prediction using the linear model for $x = 30$ is also shown on this screen, making it clear that the prediction is well outside the range of the data collected. A prediction of this kind is called *extrapolation*, and is generally a little risky. Predictions *inside* the range of the data are called *interpolation* and are often more defensible.

Other regression models

Although linear relationships are the most widely used, your *ClassWiz* allows you to explore other relationships between variables. The possible models were shown when you entered Statistics mode and can be seen again below, after tapping OPTN 1 *Select Type*.

1:Select Type	1:1-Variable	1:y=a•e^(bx)
2:2-Variable Calc	2:y=a+bx	2:y=a•b^x
3:Regression Calc	3:y=a+bx+cx²	3:y=a•x^b
4:Data	4:y=a+b•ln(x)	4:y=a+b/x

Be careful: when you choose another bivariate model from this menu, your data will stay in the calculator. But if you instead choose a 1-variable model or another bivariate model by re-entering Statistics mode with MENU 6*, the data will be deleted and you will need to enter them again.*

The models available are as follows:

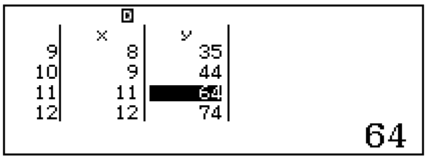
2	Linear	$y = a + bx$
3	Quadratic	$y = a + bx + cx^2$
4	Logarithmic	$y = a + b\ln x$
1	Exponential	$y = ae^{bx}$
2	Exponential	$y = ab^x$
3	Power	$y = ax^b$
4	Inverse	$y = a + b \div x$

Notice that, in almost all cases, the calculator will estimate values for a and b. The exception is quadratic regression, when a value for the quadratic coefficient c is also estimated. To interpret the calculator results, you need to make sure you know which model is being used to describe the data. Except for the first example of a linear model, all of these relationships use *nonlinear* models; that is, the model is not linear. Such models are sometimes called *curvilinear*, since their graphs show curves and not lines. When in Statistics mode, the chosen model will be displayed when you tap the OPTN or AC keys, as a helpful reminder.

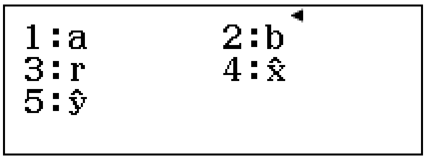
Consider another example, involving the growth of a blueberry bush. Siew-Ling planted a small blueberry bush on her birthday and measured its height to be 16 cm. She measured the height of the bush again on the same day every month for 15 months, except for a month when the family was on vacation. Here are her results:

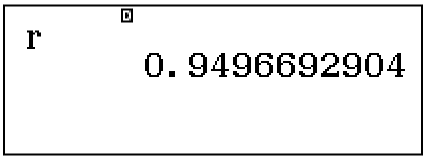
Month	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Height (cm)	16	18	19	19	24	28	31	34	35	44		64	74	77	101	105

A linear model is often a good choice for data, at least as a first attempt to represent the data. To see how well these data can be represented with a linear model, tap MENU 6 and then 2 to start a new data set, deleting previous data. Enter the months as *x* and the heights as *y,* but do not enter anything for the missing information for the tenth month, as shown below.

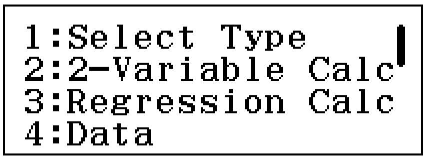


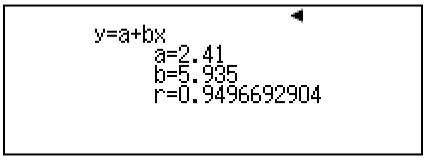
A good measure of the fit of the linear model to the data can be obtained from the correlation coefficient *r*, which in this case seems quite high, as the next screen shows, from the Regression menu:





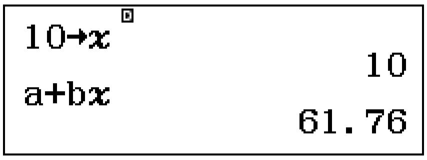
To see the details of this linear model, estimates for the *a* and *b* terms can be obtained from the Regression menu separately or from the Regression Calculation menu, shown below:



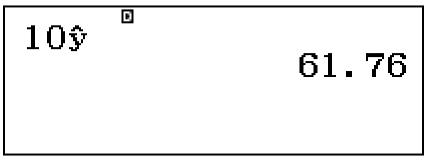


So the most appropriate linear model for these data is $y = 5.935x + 2.41$. This model suggests that the blueberry bush grows on average about 5.9 cm per month, from a starting size of a little over 2 cm.

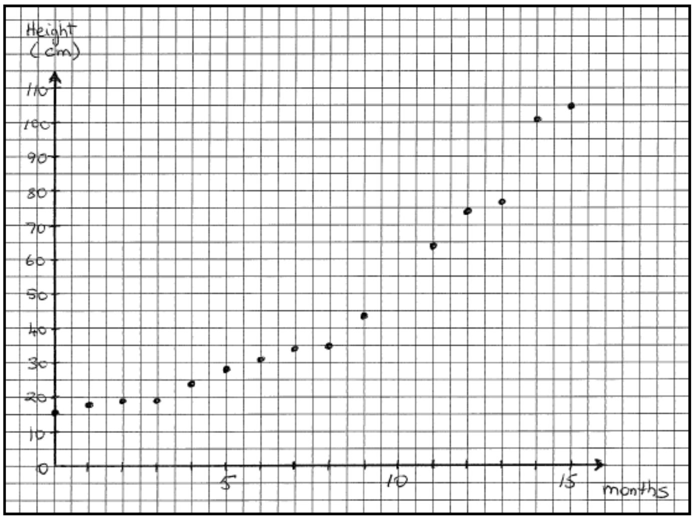
Siew-Ling used this model to predict the missing value (for $x = 10$). One way to do this is to use the model parameters, *a* and *b* obtained from the Regression menu, for the particular value of *x* (stored in the x memory) as shown below:



However, this kind of calculation can be efficiently done automatically, using the prediction capability in the Regression menu:

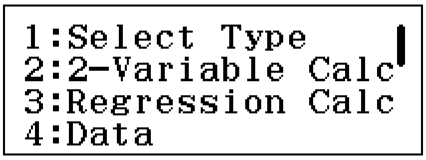


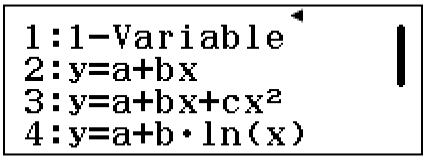
Siew-Ling was concerned that the linear model did not seem to describe the growth very well, despite the high correlation coefficient. She noticed that the actual growth seemed to be greater towards the end of the period and smaller at the start, so was not as regular as the linear model suggested. In addition, the initial size was much larger than 2.4 cm and the predicted missing value seemed too high. So she decided to draw a quick scatterplot of the data on graph paper, as shown below:



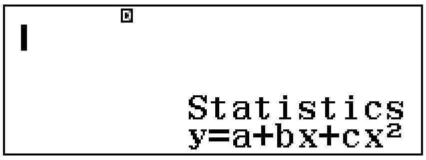
After looking at her scatter plot, Siew-Ling thought that a nonlinear relationship seemed more appropriate than a linear model, and decided to investigate a quadratic model instead.

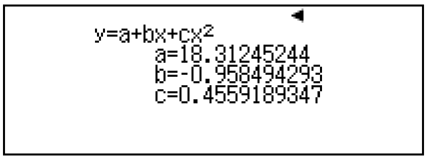
To change the model being used, it is important to not simply return and select a different model after choosing MENU 6, since this will delete all of the data, which you then will have to enter into the calculator again. Instead, tap OPTN 1 to select the *Type* of model:

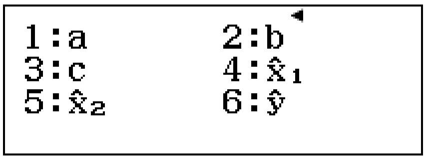




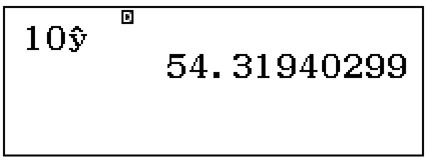
Tap 3 to select the quadratic model, and notice that the data have not been erased or changed. When you tap AC, the *ClassWiz* reminds you that a quadratic model is being used. The Regression Calculations show the best quadratic model for Siew-Ling's data. As for the linear model, the Regression menu allows you to obtain individual estimates and make predictions.



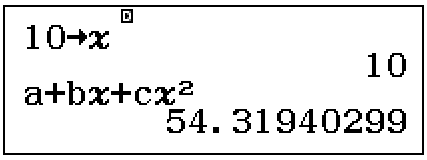




So a quadratic model for this data is $y = 18.31 - 0.96x + 0.46x^2$. The usual way of writing this is $y = 0.46x^2 - 0.96x + 18.31$, in descending order of powers of x. This model seems to fit the data better than the linear model and can be used to predict the missing value for $x = 10$:

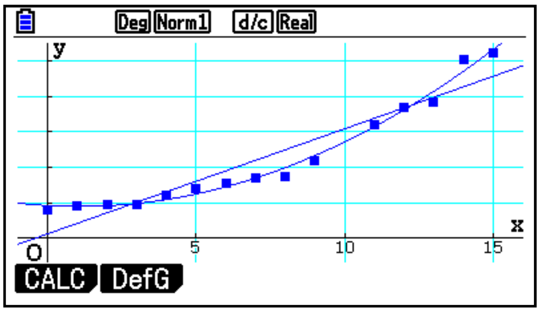


Again, to understand what the *ClassWiz* is doing here, you can make predictions like this using the regression coefficients, *a*, *b* and *c*, as shown below. As expected, the result is the same.



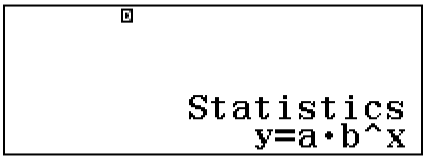
This seems like a reasonable prediction, around half-way between the height after 9 months and the height after 11 months, and seems to be a better prediction than that made from the linear model earlier.

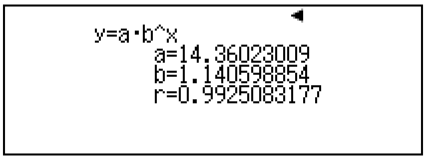
A graph is helpful to examine how well a model fits the data, although this is tedious to do by hand. A scatter plot with associated models can be obtained using a graphics calculator as shown below. Each of the best-fitting linear model and quadratic model is shown, and it is clear that the quadratic model fits the data much better, because the curve is close to most of the points, unlike the situation for the line.



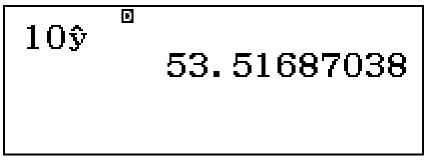
An exponential model

Siew-Ling's friend suggested that, because her data reflected natural growth of her blueberry bush, a better choice to model the data might be an exponential relationship of some kind. (Exponential functions were treated in Module 6.) So, she decided to explore this and once again used OPTN 1 to select the type of model without losing the data, this time choosing the exponential model $y = ab^x$. The results are shown below:





So the best-fitting exponential model in this case is $y = 14.36 \times 1.14^x$. Using this model, the prediction for the missing value for $x = 10$ is very close to that suggested by the quadratic model:

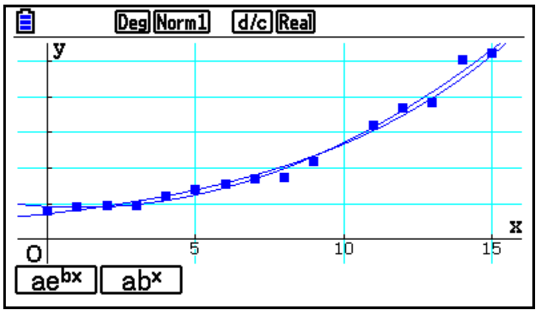


This suggests that, in this case, the two models are quite similar.

Unlike the quadratic case, the *ClassWiz* provides a correlation coefficient for the exponential model, shown above to be $r \approx 0.993$. This value is very close to 1, and, because it is larger than the value of

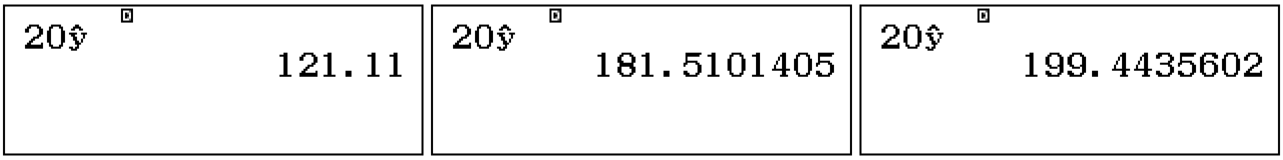
$r = 0.95$ for the linear model, it suggests that the curvilinear model fits the data better than does the linear model.

The screen below from a graphics calculator shows that the two curvilinear models used here are very close to each other over the range of data gathered.



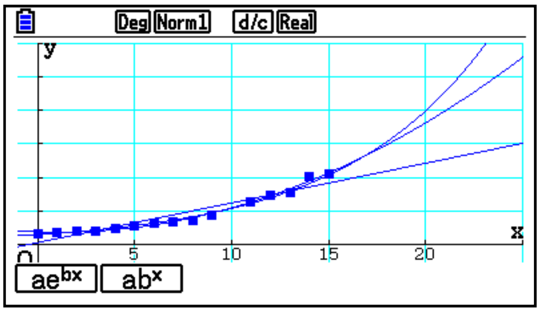
Caution is needed for making predictions using models derived from data, especially when the predictions fall outside the range of the given data. That is, it is more appropriate to make a prediction for $x = 10$ in this case (called an *interpolation*) than it is to make a prediction for $x = 20$ (which would involve *extrapolation*).

To illustrate the dangers, consider the three predictions below for the height after 20 months according to the linear, quadratic and exponential models respectively:



Concerns about extrapolation are exaggerated by having only a small set of data, too, as for this example. In general, statistical conclusions are better when more data are involved.

To see how and why the extrapolations are so different here, it is helpful to consider the following CASIO fx-CG 20 calculator screen, which shows all three models graphed at once.



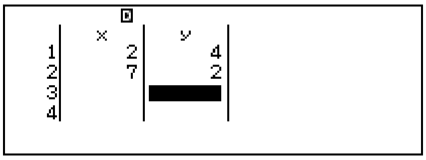
This screen shows that, while the two curvilinear models are very similar over the range of the available data, they diverge quickly afterwards and would produce large differences in estimates after, say, two years ($x = 24$). Similarly, the linear model is seen to not reflect the curvilinear nature of the growth of the blueberry plant, despite being a good fit over the range from $x = 0$ to $x = 15$.

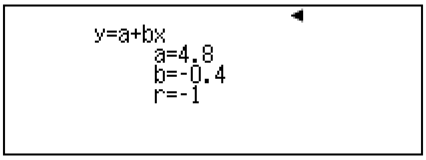
In practice, linear models are very important because they describe many relationships sufficiently well over a short period to be practically useful. Even when relationships are thought to be unlikely to be linear, linear models are still used to describe them because they are less complicated than other models and easier to use. But generally speaking, you should try to take the context of the data into account, especially to understand well the nature of relationships between variables.

A note about curve fitting

Although it is not strictly a statistical matter, you may be interested to learn that you can use the *ClassWiz* capabilities to find equations of lines and curves through points.

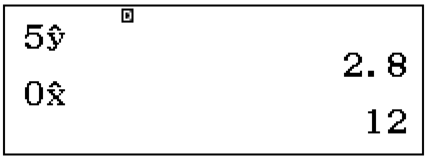
If you find a linear model for only two points, it will be the line joining the points. For example, to find the equation of the line joining the points (2,4) and (7,2) enter these as if they were data and select a linear model:



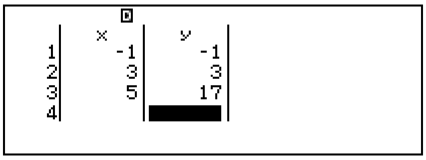


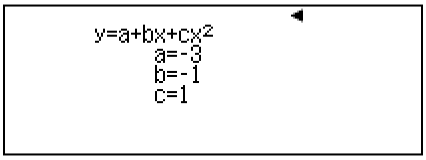
The calculator model is $y = a + bx$, so in this case the line has a slope of -0.4 and a y-intercept of 4.8. The equation of the line is $y = 4.8 – 0.4x$, which can be rearranged to give $2x + 5y = 24$.

You can determine other points on this line, using $\hat{y}$ and $\hat{x}$. The screens below show that (5,2.8) is on the line and that the line intersects the x-axis when $y = 0$ at $x = 12$.



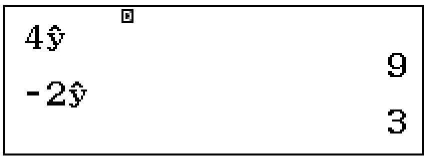
Similarly, a parabola can be fitted exactly to three non-collinear points such as (-1,-1), (3,3) and (5,17), by choosing a quadratic model and entering the points as data.



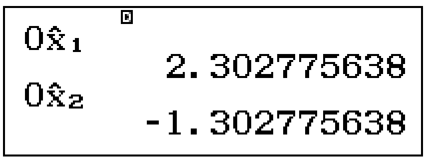


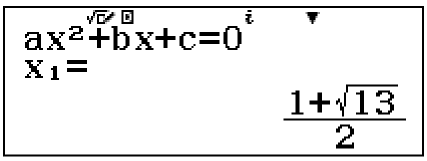
Since the calculator represents the model as $y = cx^2 + bx + a$, the parabola in this case is given by $y = x^2 – x – 3$. Substitution of values will confirm that this parabola includes all three points.

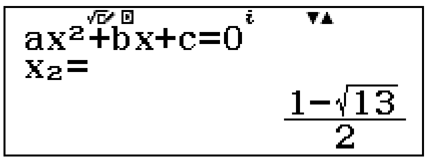
As for linear functions, you can use this model to find other points on the parabola. The screen below shows $x = 4$ and $x = -2$ and thus the points (4,9) and (-2,3)



Finding x-values for a particular y-value is a little different from the linear model, however, as more than one result is possible. The first screen below shows this, finding the two values (called x_1 and x_2) at which the parabola crosses the x-axis.







These are the two roots of the quadratic function $f(x) = x^2 – x – 3$. Check for yourself that they are approximations to the exact roots shown above in Equation mode (described in Module 6).

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. Some school students were interested in the limpet shells on the beach near their school. They selected 20 shells at random and measured the lengths and widths of the shells:

Width (x)	0.9	1.5	1.6	1.7	1.7	1.8	1.9	2.0	2.0	2.1	2.1	2.1	2.2	2.2	2.3	2.3	2.3	2.4	2.4	2.7
Length (y)	3.1	3.6	4.3	4.7	5.5	5.2	5.0	4.4	5.2	5.4	5.6	5.7	5.8	5.2	5.8	6.2	6.3	6.4	6.3	6.3

(a) Draw a scatter plot of these data by hand to study the relationship between x and y.

(b) Find the mean width and mean length of the shells.

(c) Find the standard deviation of the shell widths and lengths.

(d) Use your calculator to find the linear model that best relates the length (y) to the width (x) of the Limpet shells.

(e) Use the linear model from part (d) to predict the length of a shell that is 2.6 cm wide.

(f) Find the value of the correlation coefficient that shows the strength of the linear relationship between the lengths and the width of the shells.

(g) The students decided to see whether another model would fit the data better. Find a model of the form $y = ab^x$. Does this seem to be a better fit than the linear model to the shell data?

2. The Australian Bureau of Statistics examined the relationship between people's age and whether they accessed the Internet regularly. They obtained the following data in 2010-2011:

Age in years	15-17	18-24	25-34	35-44	45-54	55-64	65-74
% access Internet	95	96	93	90	85	71	37

(a) Make a new data table for age (using the mid-point of each interval) and the percentage of people who do *not* access the Internet.

(b) Find a linear model for the age of people (x) and the percentage who do not access the Internet (y). What is the correlation associated with this model?

(c) Use the model from part (b) to predict and interpret the Internet access of 35-year old people. Explain why it would not be appropriate to use this model to predict Internet access for children aged 4.

(d) Find an exponential model for the age of people (x) and the percentage who do not access the Internet (y). What is the correlation associated with this model?

(e) Use the model from part (d) to predict and interpret the Internet access of 35-year old people. Compare the prediction with that obtained in part (c).

(f) Which of the two models, the linear model from part (b) or the exponential model from part (d) seems to account better for the data?

(g) Draw a scatter plot of the data and compare the plot with your answer to part (f).

3. Use the calculator to find the equation of the line joining the points (1.2,3.1) and (4.6,8.2).

4. Use the calculator to find the equation of the parabola through the points (1,10), (2,11) and (5,2).

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. Relationships between human measurements can be very helpful for some purposes, such as criminal investigations or archaeological work. Examine the relationship between height and the length of the *radius* bone of the forearm (the bone from the elbow to the wrist, on the thumb side of your hand). Obtain at least 20 measurements from a range of people and use your calculator to find and use a line of best fit to predict someone's height from their radius length.

2. The following data show the population of Thailand (in millions) over recent years, according to the *CIA World Factbook*. 'Year' refers to the year number after year 2000.

Year after 2000	0	1	2	3	4	5	6	7	8	9	10
Population (m)	61.2	61.8	62.4	64.3	64.9	65.4	64.6	65.1	65.5	65.9	67.1

Draw a scatter plot and use these data to construct a suitable model for Thai population growth. Check your model with recent Thai population data. (E.g., use your model to predict the Thai population today and then check on the Internet to see how close your prediction is.]

3. Joseph likes playing computer games, and recently downloaded a new game that had 56 levels to be attained. He recorded the highest level he was able to achieve at the end of each week for ten weeks, as shown below.

Week	1	2	3	4	5	6	7	8	9	10
Highest level	2	8	11	14	15	16	18	18	20	23

When he drew a scatter plot of the data, Joseph thought that a logarithmic model might be a good choice for his learning curve. Use these data to find a suitable logarithmic model for his learning. Then use your model to make some predictions, such as what level he will reach after 20 weeks or when he might expect to reach the 40th level.

4. Sporting records lend themselves to statistical analysis. For example, the data below show the world records in 2013 for men's athletics over various distances.

Distance (m)	100	200	400	800	1500	2000	3000
Record (secs)	9.58	19.19	43.18	100.91	206.00	284.79	440.67

Use these data to construct a suitable model to predict the world record for various distances. Use your model to predict the men's world record for races of 1000 m and 5000 m. Then check your predictions with the actual records, which can be found on the Internet. Try some other sporting records of these kinds, using data from the Internet.

5. Several countries conduct projects in which students upload data about themselves to the Internet and a random sample of the data is available for download and analysis. Locate a project of interest to you of this kind, using an Internet search with *Census at school*. Download a data set in the form of a random sample of responses for 20-30 students, as well as the coding sheet that describes the data. Use your sample to analyse a suitable pair of variables (i.e., those involving numerical measurements) and compare your analysis with others.

6. Consider again the data from Siew-Ling on growing a blueberry bush. Since the scatter plot gives an impression of exponential growth, an alternative analysis is to compare the month with the *logarithm* of the height. Analyse the data again to find a *linear* model to compare month with the natural logarithm of the heights. (For example, include points (0, ln 16) and (1,ln 18) etc. in the table.) Compare your results with the exponential model in the module.

Notes for teachers

This module highlights the ways in which the *ClassWiz* can support students to think about bivariate data analysis. The module makes considerable use of Statistics mode as an important tool. The text of the module is intended to be read by students and will help them to see how the calculator can be used to analyse bivariate data. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. (b) 2.01 cm, 5.3 cm (c) $s_x = 0.40$, $s_y = 0.91$ (d) $y = 2.06x + 1.16$ (e) 6.52 cm (f) 0.90 (g) $y = 2.17\text{x}1.55^x$; this does not seem to be a much better model than the linear model. Correlation is very similar: $r \approx 0.91$ 2. (a) Tabulated data are: (16,5), (21,4), (29.5,7), (38.5,10), (49.5,15), (59.5,29), (69.5,63). (b) $y = 0.93x - 18.6$; $r \approx 0.87$ (c) $\hat{y} = 13.8$, suggesting that about 86% of 35-year olds have Internet access. This model should not be used to extrapolate to young children. (d) Use SHIFT 1 1 6 to choose an exponential model; $y = 1.68\text{x}1.05^x$; $r = 0.98$ (e) $\hat{y} = 9.2$, suggesting that about 91% of 35-year olds have Internet access. (f) The exponential model seems better, as the correlation is higher. (g) Scatterplot also suggests that exponential model is better. 3. $y = 1.5x + 1.3$ 4. $y = -x^2 + 4x + 7$.

Activities

1. It is important for students to engage in analysing real data, which is the main purpose of this activity. Emphasise careful measurements and encourage students to obtain measurements from a range of people (such as younger siblings) to improve the statistical modelling involved. The relationship is likely to be close to linear.

2. Time series data lend themselves to bivariate analysis and are well handled by the calculator. Over short periods of this kind, a linear model provides a good fit, although an exponential model would be preferred for longer periods. [Answers: y (millions) = $0.51x + 61.81$, $r \approx 0.93$]

3. Encourage students to draw a scatter plot to check that a logarithmic model appears sensible. (Choose model 4 after selecting Statistics mode.) A linear model will produce markedly different results, so that students might be encouraged to gather some data of these kinds for themselves. [Answers: y (level) = $1.95 + 8.31 \ln x$, $r \approx 0.99$. After 20 weeks, he will have reached Level 26, but it will be 97 weeks until he reaches the 40th level, according to this model.]

4. Fitting curves to data of this kind can be an interesting exercise for students, although care is needed to not over-interpret the results or use extreme precision. In this case, a linear model seems a good fit. [Answers: $y = 0.15x - 13.30$, r ≈ 0.99; predictions: 1000 m: 136.38 s (an interpolation), 5000 m: 735.08 s (an extrapolation). The records in 2013 were 131.96 secs and 757.35 secs.]

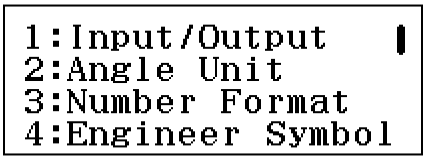
5. *Census at School* operates in many countries; links to many national sites are at http://www.censusatschool.org.uk/international-projects . You might want to check in advance and suggest a particular website and choice of variables for students to compare. Each student can download a different data set, helping their understanding of sampling variability, or they can all be provided with a single data set, depending on your preferences and levels of Internet access.

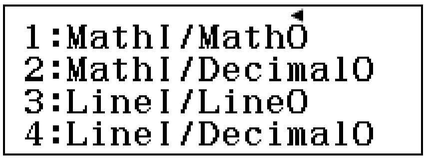
6. Log-linear analyses are a popular and powerful way of using linear regression for curvilinear data. A plot on semi-log graph paper or a plot of months vs logarithms of heights will reveal strong linearity here and provide good discussion opportunities for advanced students. [Answers: linear model is $\ln(y) = 0.13x + 2.66$ with $r \approx 0.99$, as for the exponential model. Note especially that $e^{2.66} = 14.36$ and $e^{0.13} = 1.14$, which are the parameters of the exponential model in the text.]

Module 12
Probability

Probability is an important part of modern mathematics and modern life, since so many things involve randomness. The *ClassWiz* is helpful for calculating probabilities, especially those that rely on combinatorics or those whose distribution is understood. It is also useful for simulation of events with a known probability.

In this module, unlike other modules, it is better to set the calculator to use Line output. To do this, choose [1] *Input/Output* in SET UP and then choose [3] *LineI/LineO* result format, as shown below.





Probability of an event

There are two ways of thinking about the probability of an event. When outcomes in a sample space are theoretically equally likely, you can think of the probability of an event *E* as

$$\Pr(E) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

For example, the probability of rolling a four on a standard six-sided die is 1/6, using this definition.

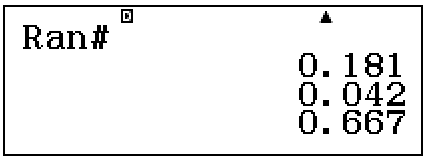
When outcomes are not known in sufficient detail to count them or to decide whether or not they are equally likely, the probability of an event is often estimated as the *long-run relative frequency* of the event. For example, suppose 10 000 batteries were tested to see if they last more than 30 days, and it was found that 7200 do last that long. Then the probability of a randomly chosen battery lasting at least thirty days is estimated to be 7200 ÷ 10 000 or 72%.

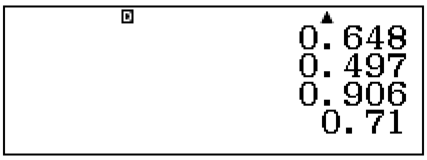
Simulating events

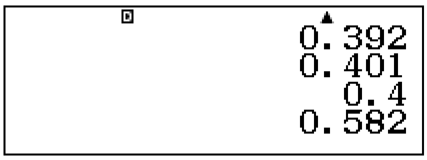
The calculator can be used to simulate random events, using the *Ran#* and *RanInt* commands.

The random number command *Ran#* simulates a random number that is greater than zero and less than 1 each time it is tapped. The numbers are uniformly distributed over the interval (0,1). So, for example, you should expect to get a number between 0 and 0.4 on 40% of the time, in the long run.

After you use the command once (with [SHIFT] [•] [=]), then tapping [=] repeatedly produces a different random number each time. The screens below show some examples. Yours will not be the same, as the numbers are random.



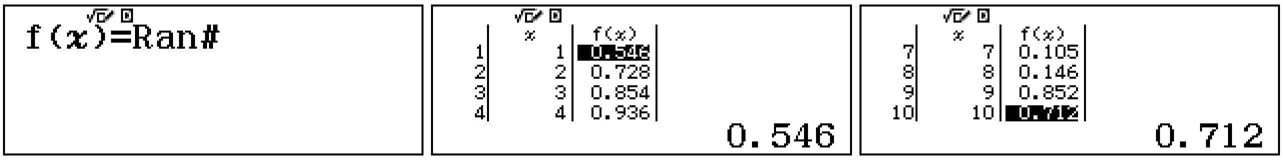




Random numbers are generally given to three decimal places, but if the final digit is zero (as in the middle screen above), the calculator reports only two decimal places.

Exceptionally (theoretically with a probability of about 1%), the final two digits will be zero, so the calculator will report only one decimal place, as the third screen shows.

As well as simulating individual outcomes (with each tap of the = key), a set of up to 30 outcomes can be simulated with a table. To do this, enter Table mode (using MENU 9) and define the function as shown below.

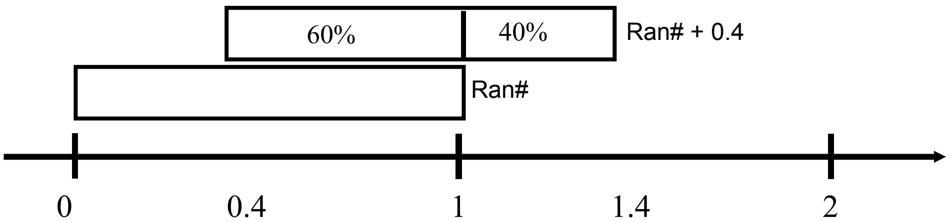


Tap = and then set *Start* = 1, *End* = 10 and *Step* = 1, each time followed by =. A table of ten random numbers (such as, but different from) those above will be generated. You can scroll up and down the values with cursor keys, first using ▶ and then using ▲ and ▼.

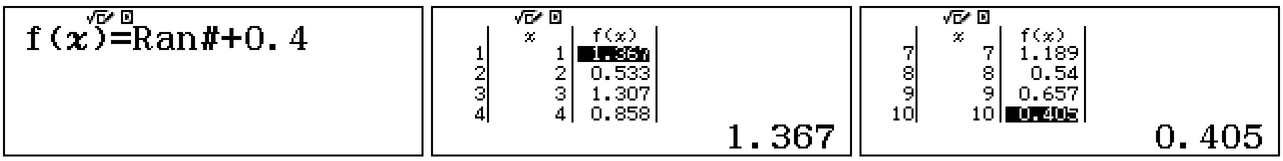
Consider again the event described earlier with a probability of 0.4. In this particular case, only three of the ten random numbers is less than 0.4, rather than the four that are expected for a probability of 0.4, but that is the nature of randomness. (To see that there were only three numbers less than 0.4, we needed to scroll through all ten values. Your values will be different from this, of course.) Remember that it is only in the long run that the results in practice will match the theoretical results.

If you regenerate the table (e.g., by tapping AC and then = three times, a fresh set of random numbers will be tabulated.

It is tedious to scroll through, read and interpret individual simulated values, so it is sometimes easier to generate numbers that are easier to identify. We think still of an event with probability 0.4 or 40%. Consider the diagram below, showing the effects of generating random numbers with the command *Ran#* + 0.4



The diagram suggests that 40% of the numbers will lie between 1 and 1.4; that is, they will start with a digit 1, while the remaining 60% will start with a digit of 0. It is easier to recognise these numbers in a table for which those starting with 1 represent a 'win' and those starting with 0 represent a 'loss'.

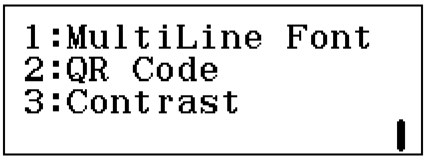


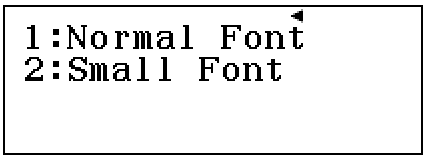
In this case, five of the random numbers were starting with a 1, so the simulation produced five successes. The purpose of using the transformation *Ran#* + 0.4 is only to make it easier to recognise and to count these numbers without errors; it does not alter the likelihood that the event will be simulated.

Simulating integers

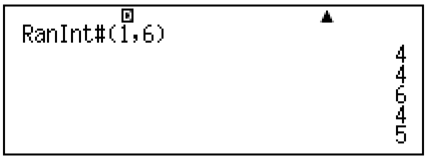
For many practical applications, it is convenient to simulate integers at random, rather than decimal numbers. For this purpose the *RanInt* (ALPHA •) command is very useful. The calculator will generate integers at random, uniformly distributed on an interval of your choosing.

To see more than one result in each screen, it is a good idea to use SET UP to temporarily shift to use the *Small* Multiline font: as shown below



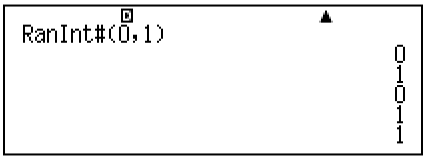


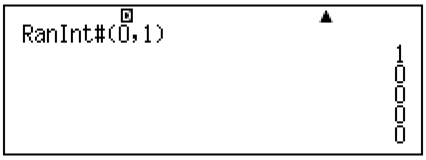
With these settings, the screens below show the generation of dice throws, with a standard six-sided die, for which each of the integers in the set{1, 2, 3, 4, 5, 6} is equally likely. The command requires a comma to separate the first and last integer. You need to tap SHIFT) to get this. Each tap of the = key produces another simulated dice throw.



In this case, the simulation produced three fours, a five and a six from five trials.

This command is useful for simulating Bernoulli processes, which can result in either of two outcomes. Provided the outcomes are equally likely, such as tossing a fair coin and counting the number of heads (0 or 1), the result can be obtained directly. Each of the screens below shows five simulated tosses of a fair coin. There were three heads in the first set, and one in the second set.





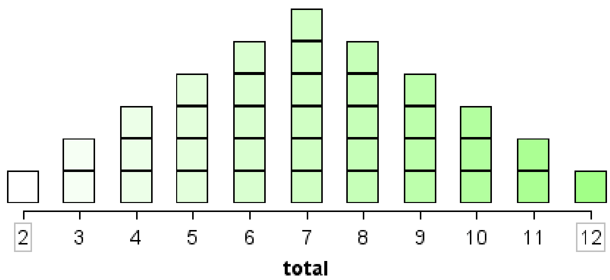
Care is needed to think about the events being simulated. For example, to simulate the rolling of a pair of standard dice, it is necessary to simulate each die separately. That is, the command used is

RanInt(1,6) + RanInt(1,6)

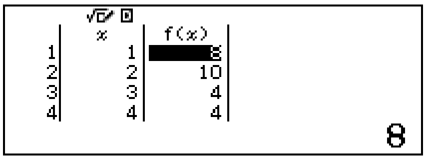
and not *RanInt(2,12)*

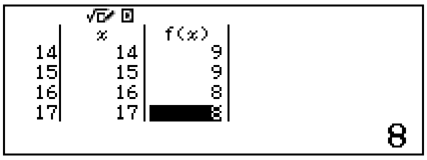
Both commands will simulate dice rolls between 2 and 12, but the second command will make it equally likely for any of the twelve possible values {2, 3, 4, … , 11, 12} to occur, which is not the case in practice. To see why this is so, study the following diagrams. The diagram on the left shows the sample space of all 36 possible outcomes of rolling two fair dice. Some results like a total of 11 are theoretically quite rare (e.g., only the results 5,6 and 6,5 produce 11), while others are likely to happen more frequently. For example, there are six different ways (shown on the diagonal) of tossing a total of 7.

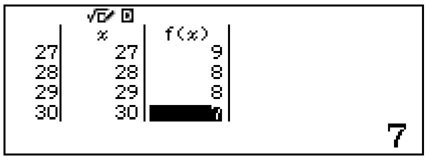
$$\begin{bmatrix} 1,6 & 2,6 & 3,6 & 4,6 & 5,6 & 6,6 \\ 1,5 & 2,5 & 3,5 & 4,5 & 5,5 & 6,5 \\ 1,4 & 2,4 & 3,4 & 4,4 & 5,4 & 6,4 \\ 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 \\ 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 \\ 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 \end{bmatrix}$$



The graph on the right shows the theoretical sample space arranged as a distribution, making it clear that the possible values for the total of 2, 3, … , 12 are not all equally likely. Long-run theoretical results will not happen in practice every time. To see this, we will generate some random data in a table. The screens below show some results from simulating thirty pairs of dice rolls correctly, using the function, $f(x)$ = *RanInt(1,6)* + *RanInt(1,6)*. *Start* at 1, *End* at 30 and make the *Step* 1.







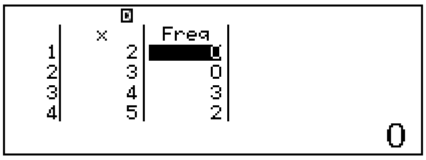
In this case, we counted all 30 outcomes and obtained by hand the following distribution:

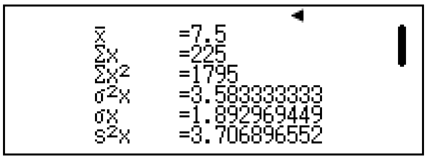
Result	2	3	4	5	6	7	8	9	10	11	12
Frequency	0	0	3	2	3	5	9	4	3	0	1

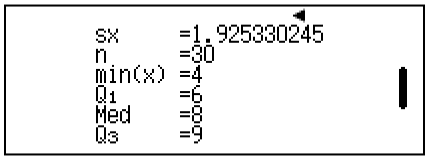
As expected, this is not the same as the theoretical distribution shown above, which shows what will happen in the long run theoretically. Each time a set of dice rolls is simulated, a different result will be obtained. For example, here is another set of 30 simulated dice tosses:

Result	2	3	4	5	6	7	8	9	10	11	12
Frequency	0	2	2	4	4	3	6	5	1	1	2

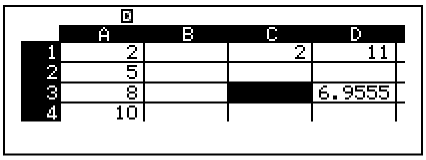
Results like these can be analysed in Statistics mode, which is discussed in Module 10. For example, here is a brief summary of the first set of 30 tosses:

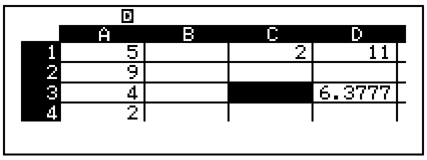


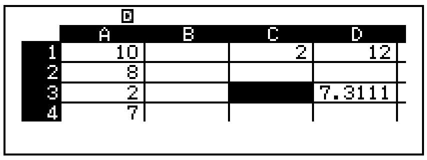




Generate your own data and compare them with these to see differences and similarities. Module 13 shows you how to simulate and analyse 8-sided dice rolls, using the *ClassWiz* spreadsheet, which is more efficient than a table for this purpose. The screens below show three separate examples of this for a pair of 6-sided dice. In each case, the results are in column A, the mean total of 45 rolls is shown in cell D3 and the minimum and maximum totals are shown in C1 and D1 respectively.







Variation is evident in these results. For example, in neither of the first two simulations were any totals of 12 obtained, and the mean scores vary from 6.4 to 7.3. Such is the nature of randomness. Compare these results with your own simulations and with those of others.

Combinatorics

Combinatorics is concerned with systematic counting of things. It is important to be able to do this in order to determine theoretical probabilities in many practical situations. The numbers involved in situations are often very large, and so computation with a calculator is often necessary.

For example, the number of ways in which a set of different objects can be arranged in order – which is known as the number of *permutations* – often involves *factorials* of numbers. Here is a small example:

Three children finish a running race. If there are no ties (equal places), in how many different orders can they finish?

In this case, the different possibilities can be listed. If the three children are represented by A, B and C, the complete set of six possibilities is shown below.

ABC ACB BAC BCA CAB CBA

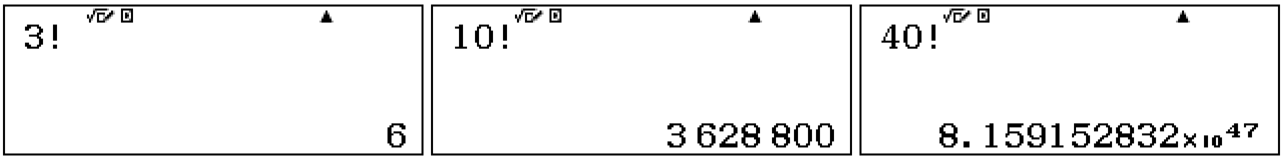
This problem could be analysed by noting that there are three different possibilities for first place; once first place is determined, for each possible first place there are two possibilities for second place; once the first two places are determined, there is only one possibility left for third place. So the total number of orders is:

$$3 \times 2 \times 1 = 6.$$

This analysis applies in general, so the number of orders or permutations of n distinct objects is given by n factorial, which is

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

Factorials are often very large numbers, much too large to calculate by hand. So the calculator has a command $x!$ (SHIFT x^{-1}) for this purpose:



The screens show that there are more than 3.6 million different orders in which ten students could finish a race and almost 10^{48} different orders of 40 students finishing a race. The calculator cannot compute factorials larger than 69!, as they are too large to fit in the calculator, which is restricted to numbers less than 10^{100}.

In some situations, we are interested in the number of orders possible, but from a restricted set. For example, if ten children enter a race, how many possible results are there for first, second and third place (once again assuming no ties)?

Using the same logic, there are ten choices for first place, after which there are nine choices left for second place and finally eight choices for third place. Altogether, the number of different permutations is

$$10 \times 9 \times 8.$$

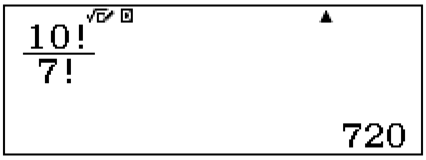
This can be thought of as

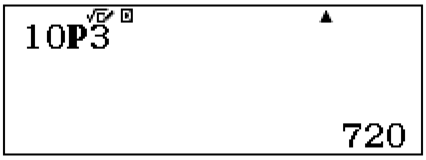
$$\frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1} = \frac{10!}{7!}.$$

The mathematical symbol for this result of permuting ten objects, three at a time is ${}_{10}P_3$. In general, the same logic suggests that the number of permutations of n objects taken r at a time is

$$n\mathrm{P}r = \frac{n!}{(n-r)!}.$$

In practice, calculations can usually be done in full, but it is easier and quicker to use the permutations command nPr on the calculator, via SHIFT ×. Notice that you enter the value of n, followed by the command, followed by the value of r.





The screens make it clear that there are 720 different permutations or orders in which the three places can be filled from only ten students; many people are surprised that this number is so large.

A third kind of combinatorics, or counting, problem is to count the number of combinations, regardless of order. In this case, we might be interested in counting how many sets of three place-getters are possible from a set of ten students, but are disinterested in who is first, second or third. This is defined as the number of *combinations* of ten students, taken three at a time and defined as

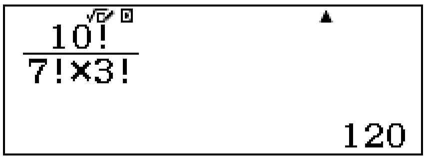
$$10C3 = \frac{10\,\mathrm{P}\,3}{3!} = \frac{10!}{(10-3)!3!}.$$

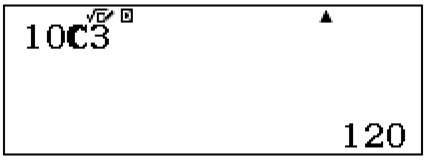
because once a set of three is chosen, we don't want to count all 3! ways of permuting them.

In general, the number of combinations of n objects taken r at a time is

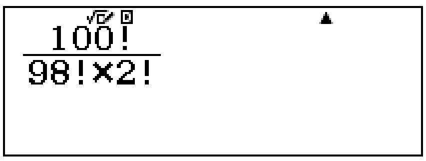
$$nCr = \frac{n!}{(n-r)!r!}.$$

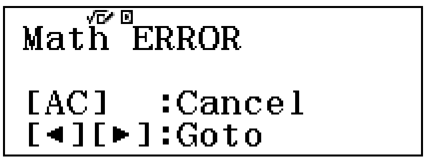
The screens below show this value determined directly and using the calculator combinations command $n\mathrm{C}r$ with SHIFT ÷:



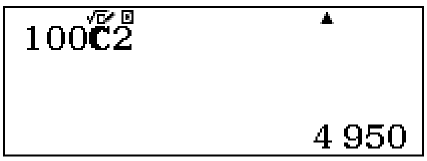


An advantage of using the $n\mathrm{C}r$ and $n\mathrm{P}r$ commands is that they permit larger numbers to be used in calculations. For example, if 100 people are in a room and all shake hands with each other, the number of handshakes is ${}_{100}\mathrm{C}_2$. As the screens below show, the numbers involved (100! and 98!) are too large for the calculator to handle, resulting in the Math error.



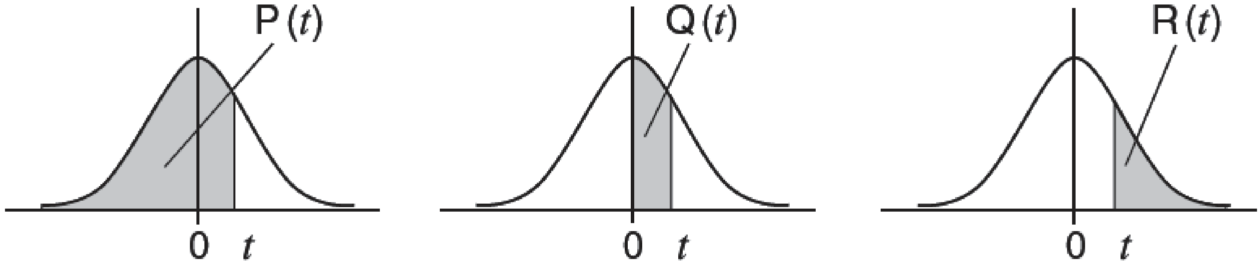


The task is not problematic with the combinations command, however, as the following screen shows.



The Normal probability distribution

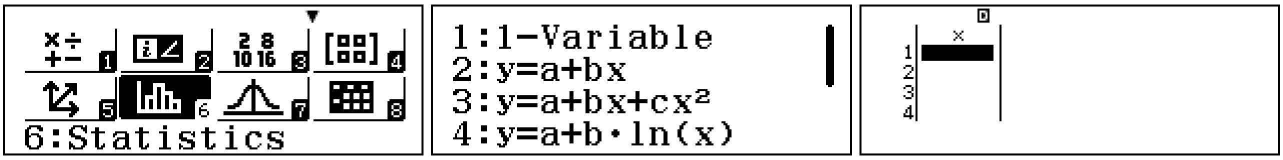
Many naturally occurring data follow a *normal probability distribution*, with a characteristic bell-shaped curve shown below. So it is useful to have access on the calculator to the standard normal distribution (i.e. a distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$).



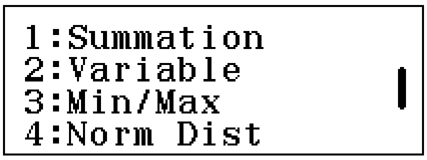
The total area under the normal distribution is 1 and the probability that a standard normal random variable z will take values in a particular range is given by $P(t)$, $Q(t)$ and $R(t)$ as shown above. These three values refer to the following probabilities:

$P(t) = \text{Prob}\ (z \le t)$ $\quad Q(t) = \text{Prob}\ (0 \le z \le t)$ $\quad R(t) = \text{Prob}\ (z \ge t)$

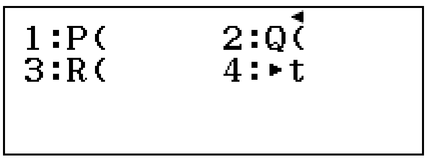
To access these normal distribution commands, you need to set the calculator into 1-variable Statistics mode, using MENU 6 1, as shown below.



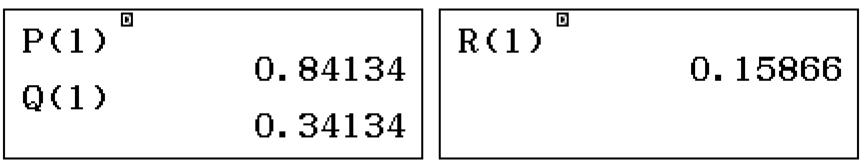
The calculator is now ready for entering univariate statistics, as described in detail in Module 10, but our interest in this module is not in data analysis but only in the use of the normal distribution, assuming that we already know the mean and the standard deviation of the population of interest. So tap AC and then OPTN ▼ to see the following screen:



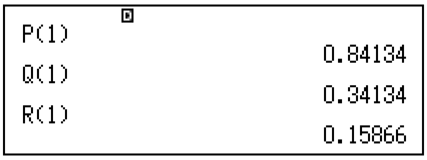
Finally, tap 4 to obtain the Normal distribution commands, which are shown below:



These commands are related, as can be seen by finding the respective probabilities for a particular value. For example, consider the value of $z = 1$. As the diagrams above suggest, $Q(1) = P(1) - 0.5$ and $R(1) = 1 - P(1) = 0.5 - Q(1)$. You can enter these probabilities as they are written to check:

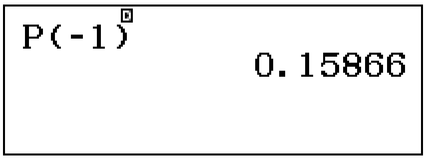


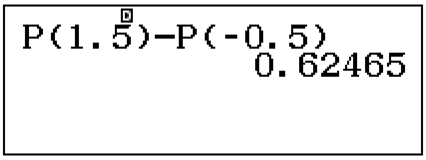
Notice that if you prefer the Multiline Font in SET UP to be set to *Small Font* size, only one screen is needed, although the results are the same:



The three probabilities associated with $z = 1$ reflect these relationships.

As the normal distribution is symmetrical about $z = 0$, Prob $(z \leq -1)$ = Prob $(z \geq 1)$. The first screen below shows this relationship and also demonstrates that both positive and negative values for z can be used, unlike the usual situation for printed tables for the normal distribution.





The second screen shows that probabilities for an interval can be obtained directly. For example, to find Prob $(-0.5 \leq z < 1.5)$, use the calculator to find Prob $(z \leq 1.5)$ – Prob $(z \leq -0.5) = 0.62465$.

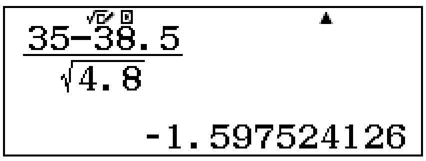
In practice, most variables are not distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, and a suitable transformation must be made for the tabulated values to be used. In general, if a variable X is distributed normally with mean μ and variance σ^2, then the transformed variable

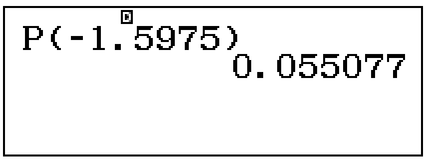
$$z = \frac{X - \mu}{\sigma}$$

will be a standard normal variable. (In Module 10, the calculator use of t instead of z is described.) Probability calculations can be used directly with the transformed variables. To illustrate:

> *A machine produces a quantity of curry sauce to be sealed into plastic bags for packaging and sale. Suppose the machine is known to produce masses (in grams) of sauce that follow a normal distribution with mean $\mu = 38.5$ and variance $\sigma^2 = 4.8$. The company producing the sauce has printed packages that claim to contain 35 g of sauce.*
>
> *How likely is it that a randomly chosen package will have less than this amount?*

The z value associated with $X = 35$ is $z = \dfrac{35 - \mu}{\sigma} = \dfrac{35 - 38.5}{\sqrt{4.8}} \approx -1.5975$, as shown below (in Calculation mode):

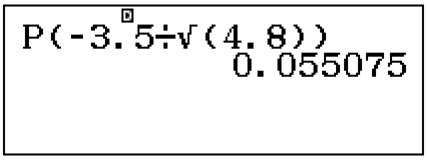




So the probability can be found by using this value in Statistics mode with $P(z)$, as shown at right above.

The result of about 0.055 shows that a little more than 5% of the packets produced will have a smaller mass of sauce than is claimed.

It is inconvenient to switch between Statistics mode and Computation mode for calculations like this. So it may be preferable to do all of the calculations within Statistics mode, as shown below.



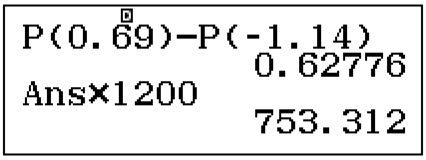
Notice that, in Statistics mode, the fraction key does not allow fractions to be entered as natural expressions, so a division statement may be used here instead. Notice also that the result is slightly different from the earlier result as the z value has not been rounded to four decimal places.

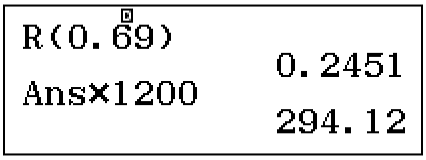
Various other probability calculations are possible with the information provided about the sauce machine. For example, suppose a batch of 1200 packets is produced in a day. How many packets will have between 36 g and 40 g of sauce? How many will have more than 40 g of sauce?

$$\text{Prob }\left(36 \le X \le 40\right) \quad = \text{Prob}\left(\frac{36-38.5}{\sqrt{4.8}} \le z \le \frac{40-38.5}{\sqrt{4.8}}\right)$$

$$\approx \text{Prob } (-1.14 \le z \le 0.69)$$

These z values can be entered directly into the calculator:



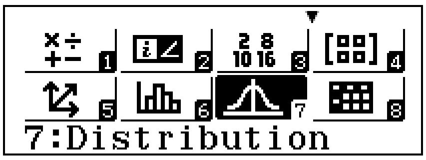


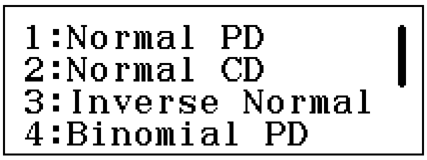
The screens show that the probabilities are about 62.8% and 24.5% respectively. As there are 1200 packets produced daily, these percentages can be used to predict that about 753 of the 1200 packets will be expected to have between 36 g and 40 g of sauce, while about 294 of the 1200 packets are expected to contain more than 40 g of sauce.

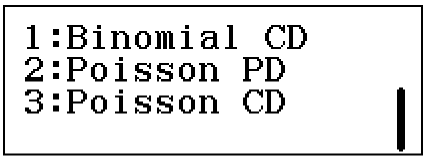
The use of the normal probability distribution in this module assumes that you already have access to the mean and standard deviation for the situation of interest. Module 10 deals in detail with using the normal distribution when you have the primary data rather than these summary statistics.

Using distributions

In addition to the Normal distribution functions in univariate statistics, the *ClassWiz* provides access to three key probability distributions in *Distribution* mode, accessed with MENU 7:







For each of the Binomial, Normal and Poisson distributions, point distribution (PD) calculations and cumulative distribution (CD) calculations are available, as suggested by the screens above.

After selecting a type of distribution, you need to enter detailed parameters for the distribution and will then be able to calculate various probabilities. To return to the above screen to select a different probability distribution, tap OPTN and then 1.

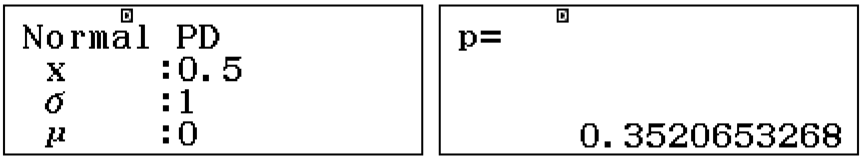
We will firstly look again at the Normal distribution calculations, this time in Distribution mode.

Normal distribution calculations

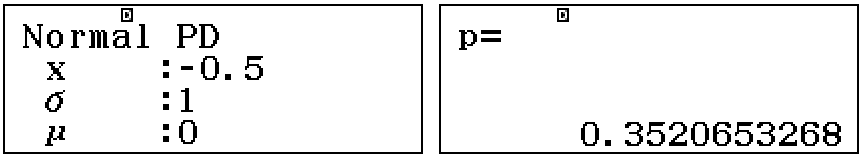
After choosing a type of Normal distribution calculation, you first need to enter the mean (μ) and the standard deviation (σ) of the Normal distribution of interest, followed by = in each case. In each of the three cases, you will need to enter other information as well, in the same way.

When you tap = without entering data, the calculator will undertake the calculation. Tapping = again will return you to the list to enter different parameters if you wish.

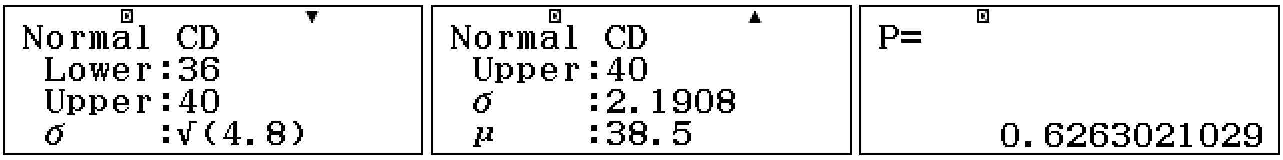
The *Normal PD* calculation provides the probability density of the distribution – the height of the Normal distribution graph at a particular value of the variable. This is useful for sketching the graph and for studying its properties. Consider the example below for the standard Normal distribution (i.e. μ = 0 and σ = 1) showing the height of the graph when $x = 0.5$.



Tap = to enter new values. As shown below, the height of the curve when $x = -0.5$ is the same, because the distribution is symmetrical around the mean of 0:

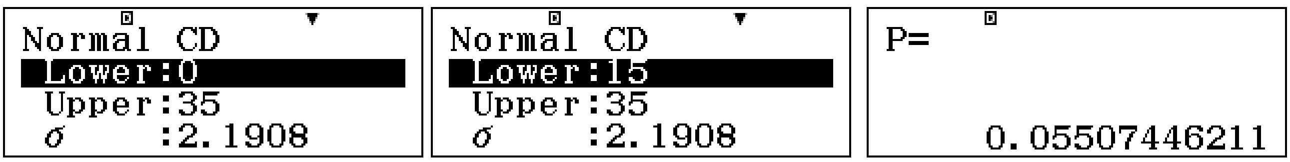


The Normal CD (cumulative distribution) function allows you to find the probability that a randomly selected element falls in a certain interval, as for the example of the curry sauce described earlier in this module. The screens below show how to use the function. Notice that it is not necessary to calculate σ before entering it as the calculator will evaluate the square root of the variance.



This result suggests that 62.6% of the sauce packets will have between 36 and 40 g of sauce. This is close to the earlier result (which was rounded, unlike this one.)

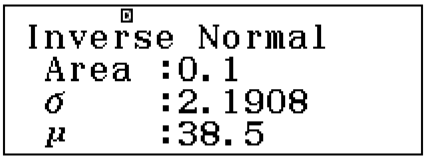
If you wish to find normal probabilities at the extremes of the distribution, enter suitably small or large values for the Lower and Upper settings. For example, to determine the probability that a curry packet will have less than 35 g of sauce (the amount guaranteed in writing on the packet), it is necessary to determine Prob ($x < 35$). Since most values of the normal distribution are within three standard deviations of the mean, any suitably small value will suffice for the Lower setting:

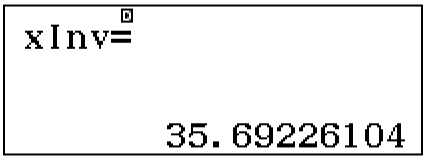


In this case, both a setting of 0 and of 15 produce the same result, which also matches the result obtained earlier in this module.

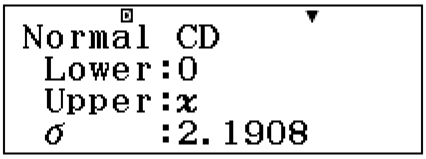
The *Inverse Normal* distribution allows you to determine a value of the variable associated with a particular probability. This useful facility is not available directly in the univariate statistics Normal distribution commands.

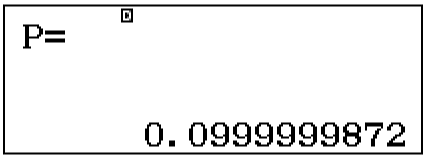
For example, consider again the curry sauce packets. To find the mass of sauce in the bottom 10% of packets, we need to find the value of x for which the area under the Normal curve is 0.1. The commands below show how to enter this calculation into the Inverse Normal calculator:





It seems that 10% of the packets have no more than 35.7 g of sauce (or, put another way, 90% of packets have at least 35.7 g of sauce). You can check what this means by either writing down the value or storing it to a memory (using [STO] [x]) and then using the Normal CD calculation, as shown below. The result is (very close to) 10%, as expected.

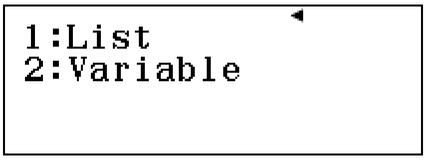


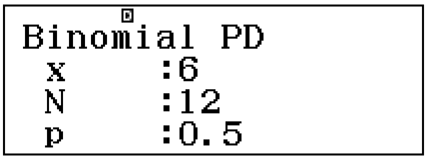


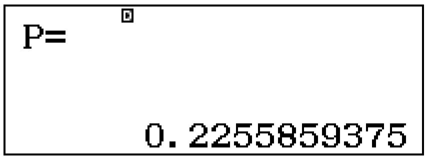
Binomial distribution calculations

The Binomial distribution is based on a Bernoulli process (i.e., a result of 0 or 1 with a constant probability) that has been repeated many times. It is often used with a series of independent Bernoulli events with only two outcomes that are described as 'win' or 'loss'. A good example involves tossing 12 times a coin for which the probability of a head is $p = 0.5$ each time. The Binomial distribution will provide the probability of various numbers of heads occurring.

When *Binomial PD* is chosen, you must then choose between a list and a variable, as shown below. A variable allows for one probability to be found, while a list allows you to enter a list of values of interest. The example below shows the Variable option being used to find the probability of obtaining exactly $x = 6$ heads in $N = 12$ tosses with the probability of each head is $p = 0.5$.

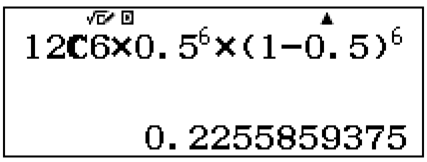






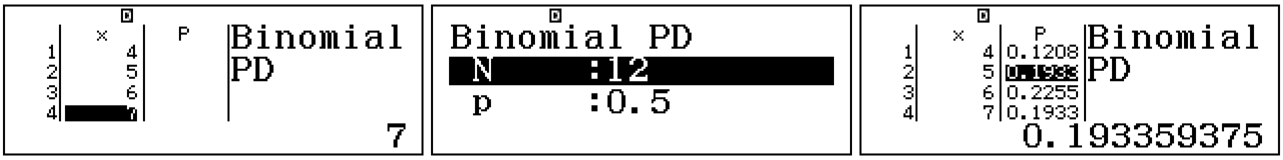
Some people are surprised that 12 coin tosses produces six heads less than a quarter of the time, as they intuitively think it should occur more frequently than that.

The probability evaluated is in fact Prob (x) = ${}_N C_x \times p^x \times (1-p)^{N-x}$, as the screen below shows, but it is clearly easier to use the application than to enter the calculation by hand.



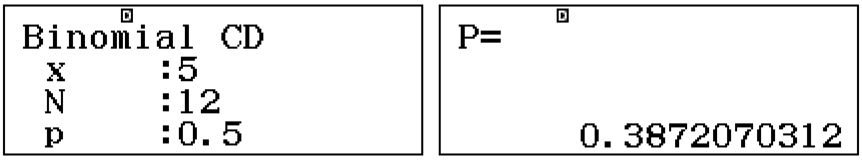
There are many (in fact 12) other possibilities when 12 coins are tossed. Tap [=] to evaluate another. Entering a list of values of interest allows you to find the probability for several values of x efficiently. To do this, you need to return with [OPTN] [1] [4] and select *List* instead of *Variable*.

In the screen below, we have chosen to find the probabilities of 4, 5, 6 or 7 heads.

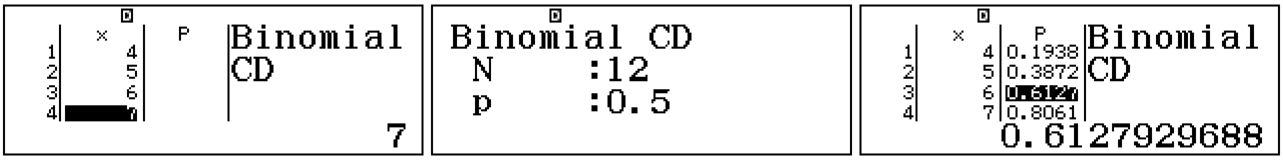


You can see that there is the same probability of obtaining 5 and 7 heads, and each is almost as likely as obtaining six heads. This particular distribution seems to be symmetrical.

The *Cumulative Binomial* distribution provides the probability that the number of wins will be less than or equal to the chosen value of x. So, in the screen below, the probability that the twelve coin tosses result in 0, 1, 2, 3, 4 or 5 heads is found to be about 38.7%.

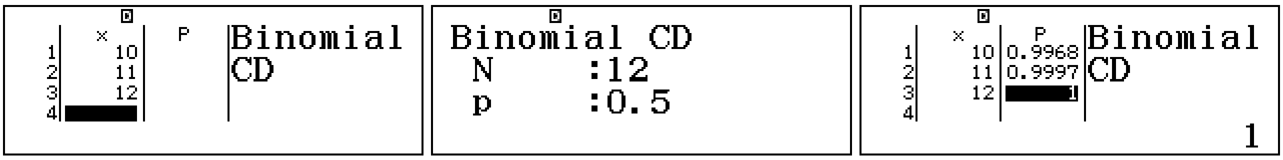


The list option again allows you to obtain cumulative probabilities for various values of x:



Notice that about 61% of the time there will be no more than six heads in twelve tosses.

As you would expect, since the maximum number of heads possible in 12 tosses is 12, notice that the cumulative binomial probability for 12 is 1:



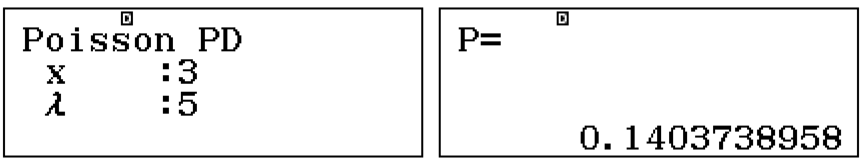
Poisson distribution calculations

The *Poisson* distribution is used to find the probability of a number of discrete events occurring in a certain interval, when it is known that they occur at a particular rate. For example, if a help desk gets 5 requests per hour on average, how likely is it to get only three requests in the next hour?

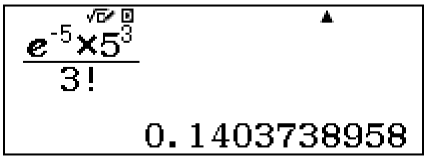
Unlike the Binomial distribution, which has two parameters (N and p), a Poisson distribution has only one parameter, the mean rate per interval, represented by the Greek letter λ (lambda). When events follow a Poisson distribution, the probability of x occurrences in the interval is given by:

$$\text{Prob}\,(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Select *Poisson PD* and enter the help desk data as below in the *Variable* option:

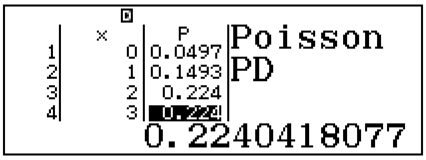


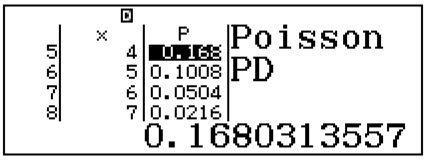
Again, you can verify that this calculation is correct by doing it manually in Calculation mode; the result is the same:

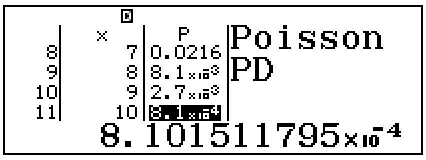


The probability shows that the help desk will get only 3 requests in an hour about 14% of the time.

As for the Binomial distribution, sometimes a list of Poisson probabilities is preferred, rather than a single value. For example, if raisins are distributed into a small cake mix in such a way that each small cake will get on average three raisins, we may wish to know how likely various possibilities are. In this case the mean rate is λ = 3. To determine several probabilities at once, a *List* is a good choice:

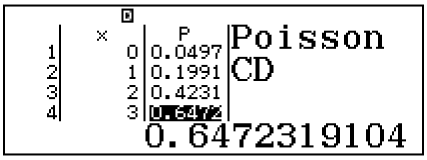






It seems that two or three raisins is the most likely outcome, occurring about 45% of the time. Notice that it is extremely unlikely, but it is not impossible, to get 10 raisins in a cake.

As for the Binomial Distribution, we may sometimes be interested in a *Cumulative Poisson* distribution. For example, how likely is that a cake will have no more than three raisins? Such questions can be answered in the Poisson CD application, as shown below. A *list* of values has been entered to allow for a closer study of possibilities.

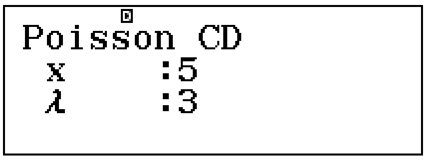


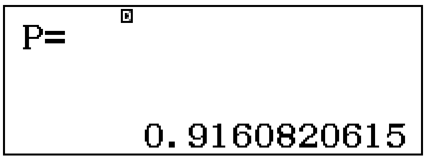
Assuming the Poisson distribution applies, these values suggest that a cake will have up to (but no more than) three raisins around 65% of the time.

Sometimes, our interests are in the opposite direction. For example, suppose that the cake mix manufacturer may wish to know the probability that a cake will have six or more raisins. To answer such questions, it is still necessary to use the Poisson CD application, but we first need to note that the number of raisins x must be an integer and that

$$\text{Prob}\ (x \geq 6) = 1 - \text{Prob}\ (x \leq 5).$$

We can use the calculator to determine Prob ($x \leq 5$) ≈ 0.92, as shown below.





Then, Prob ($x \geq 6$) = 1 – Prob ($x \leq 5$).≈ 1 – 0.916 = 0.084.

That is, only about 8% of cakes will have six or more raisins.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. Use your calculator to simulate ten random numbers between 0 and 1.

2. Use Table mode to simulate a set of 30 random integers between 1 and 8.

3. A set of random numbers was simulated in a table using the command $f(x) = 100Ran\#$. Describe what kinds of numbers you will expect to obtain in this way. Then use the calculator to check if your prediction is correct.

4. The gambling game of Lotto requires players to choose six different numbers from 1 to 45. Kai Fai wanted to use his calculator to simulate a suitable set of Lotto numbers.
   (a) Describe how to use a Table to do this.
   (b) Explain why this is sometimes an unsuccessful strategy for generating suitable numbers.

5. Evaluate (a) 8! (b) 14!

6. ${}_7C_5 = \dfrac{7!}{5!2!}$. Use your calculator to evaluate these expressions to decide which of these, if any, gives the correct answer:

   (a) 7! ÷ 5! x 2! (b) 7! ÷ (5! x 2!) (c) $\dfrac{7!}{5! \times 2!}$ (d) 7C5

7. There are 22 horses in a race. In how many ways could the first three places be filled?

8. Luke is making pizzas. Each pizza will have four toppings. He has ten toppings altogether, from which to choose. How many different pizzas can he make?

9. There are 70 students at a field camp. The leader wants to choose five students to lead field groups. In how many ways could she choose the set of leaders?

10. If z refers to the standard normal distribution (with mean of 0 and standard deviation of 1), use the calculator in Statistics mode to evaluate:

    (a) Prob ($z < 0.74$) (b) Prob ($z > 1.8$) (c) Prob ($-1.5 < z < 0.6$)

11. The weights of koala bears in Australia are known to be normally distributed with a mean of 11 kg and a standard deviation of 1.2 kg. What is the probability that a randomly selected koala bear would weigh more than 12.5 kg?

12. The heights of young women in the USA are normally distributed with a mean of 162.6 cm and a standard deviation of 6.8 cm. If a young US woman is selected at random, what is the probability that she is between 158 cm and 165 cm high?

13. A fair coin is tossed 10 times. Find the probability of
    (a) getting exactly three heads (b) getting more than six heads.

14. Over time, a Call Centre gets an average of 4.2 calls per minute. Find the probability that, in the next minute, it will get
    (a) only one call (b) more than six calls

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. A regular tetrahedron is a pyramid with four faces, each of which is an equilateral triangle. A die in this shape has faces marked with 1, 2, 3 and 4 spots.
   (a) If this tetrahedron is tossed at random, what is the probability that the number of spots on the down face is 3?
   (b) Use your calculator to simulate 20 tosses of two of these tetrahedral dice and find the sums of the down facing sides. Which sum occurred most often?
   (c) Repeat step (b) and compare results with your first attempt and with other students.
   (d) Which number do you expect to get most often?

2. A "Pick 4" lottery in the USA announces a 4-digit number each day; players win if the number they have chosen matches the winning digits. Suppose you choose 3297.

   (a) What is the probability that your number matches the winning number in the correct order?

   (b) What is the probability that your number matches the winning number in any order?

   (c) Simulate 30 choices of number by the Lottery. If your chosen number is still 3297, did you win?

3. For the country you live in, find out the distribution of heights of adult males and females. Assuming that the heights are normally distributed,

   (a) find the probability that an adult male selected at random is between 165 cm and 175 cm in height.

   (b) find the probability that an adult female selected at random is between 160 cm and 165 cm in height.

   (c) Compare probabilities in (a) and (b) with your own observations and personal experience about the heights of men and women in your country.

4. A professional basketball player is 85% successful in shots from the free throw line. If she has six free throws in a game, how likely is she to be successful with all six? What is her most likely score? How often will she be successful with no more than four shots? Investigate what happens if more shots are taken or her success rate changes.

5. A group of six friends were comparing birthday star signs (of which there are twelve spaced over the year according to birthdays). How likely is it that at least two of the friends will have the same star sign? Use a table on your calculator to simulate the star signs; do this several times and compare your results with other people to see how closely the simulated data match your expectations. What would happen if the group contained more people?

6. A major city intersection is known to be a source of many accidents and city officials have documented that there are 2.9 vehicle collisions on average at the intersection per month. In one month, five collisions were reported; how many times per year is this likely to happen by chance?

   Investigate the consequences of this collision rate. For example, would you expect there to be a collision-free month in the course of a year? What is the most likely number of collisions per month? What would be the consequences of the collision rate changing down to 2.5 or up to 3.5 collisions per month?

Notes for teachers

This module highlights the ways in which the *ClassWiz* can support students to learn about randomness, probability and systematic counting, as well as to use the calculator to generate random data and use key probability distributions. Both Statistics and Distributions mode are used. The text of the module is intended to be read by students and will help them to see how the calculator can be used to deal with matrices in various ways. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. Use *Ran#* (SHIFT ·) and tap = ten times 2. Use *f*(*x*) = *RanInt*(1,8), *Start* =1, *End* =30, *Step* =1 3. Numbers will start with a number from 0 to 99 4. (a) Use *f*(*x*) = *RanInt*(1,45), *Start* =1, *End* =6, *Step* =1 (b) Sometimes two of the six random integers will be repeated 5. (a) 40 320 (b) 87 178 291 200 6. (a) 84 (b) – (d) 21 (which is correct) 7. ${}_{22}P_3 = 9240$ 8. ${}_{10}C_4 = 210$ 9. ${}_{70}C_5 = 12\ 103\ 014$ 10. (a) P(0.74) ≈ 0.770 (b) R(1.8) ≈ 0.036 (c) P(0.6) – P(-1.5) ≈ 0.659 11. 0.106 12. 0.389 13. (a) 0.117 (b) 1 – 0.828 = 0.172 14. (a) 6.3% (b) 1 – 0.867 = 0.133

Activities

1. Students should use their calculators to generate data at random using *RanInt* (1,4) and record the results in a table on paper. Encourage them to repeat the exercise at least once and to compare results with others, in order to see that random variation occurs. [Answers: (a) ¼ (d) The most likely result of adding a pair is 5; this will occur most often, but many sets of 20 will be needed.]

2. Parts (i) and (ii) require some analysis of the situation, which might be done as a whole class if the ideas are unfamiliar to students. Simulations are unlikely to produce ‘winners’ as the probability of an exact match is only 1/10000 and if order is ignored, 24/10000. This experience may help students to see how unlikely they are to win games of this kind or at least help them to start analysing them and using their calculator to simulate results.

3. This activity will allow students to explore a real situation using their calculator. Data on adult heights can be found at various sources on the Internet, such as official sources or at *Wikepedia:* http://en.wikipedia.org/wiki/Human_height#Average_height_around_the_world . Data on variance of heights may need to be estimated; a typical figure is about 7.5 cm for both males and females. [Answers: Results vary from country to country. In Australia, male mean height is 178.4 cm and female mean height is 163.4 cm; the resulting probabilities are 29% and 26% respectively, suggesting that around one quarter of adult heights are in the given ranges.]

4. This activity involves students using the Binomial distribution with $N = 6$ and $p = 0.85$ and might provoke a discussion about over-interpreting players’ streaks of successes or failures. Encourage students to experiment with other probabilities or, if data are available, to model the performance of real local players. [Answers: 0.38, 5, 0.22]

5. This is a variation on the classical Birthday Problem. Assuming each of the twelve star signs is equally likely simulate these with *RanInt*(1,12). A table with six entries will simulate the process effectively. Encourage students to conduct several simulations and to record and compare the results. [Answer: analysis will reveal that the probability of at least one match is 1343/1728 ≈ 78%, so simulated data will reflect this. A larger group will increase the likelihood of a match.]

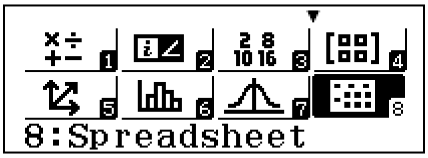
6. This activity involves students using the Poisson distribution with $\lambda = 2.9$. Encourage them to experiment with other rates or even to check for data regarding a local intersection. [Answers: Prob = 0.094, so typically one month per year; Prob = 0.055, so expect one collision-free month in two years. The most likely number of collisions is 3, but 2 is almost as likely.]

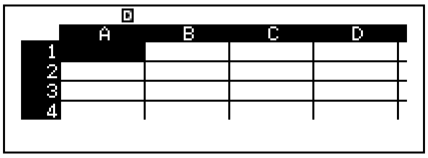
Module 13
Modelling with spreadsheets

In this module, the *ClassWiz* spreadsheet will be explored first to understand how it works and how it can be used for various mathematical purposes. Then, we will consider some examples of mathematical topics that are well handled by using a spreadsheet.

Getting started

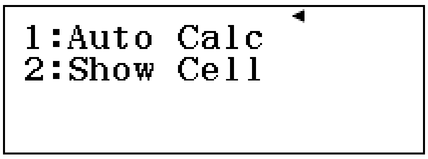
The *ClassWiz* spreadsheet comprises a rectangular array of 225 cells, with five columns labelled A to E and 45 rows labelled numerically. Enter spreadsheet mode with MENU 8 *Spreadsheet* to see a blank spreadsheet as shown below. Any information previously in the spreadsheet will be cleared when you enter Spreadsheet mode (which is important to remember when you are using it, as changing to another mode will lose all the information).

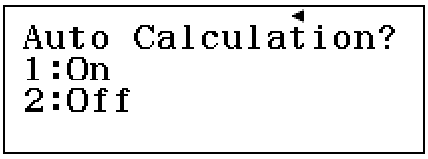


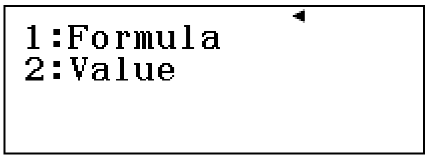


Each cell has an *address*, like a grid reference on a map, with the letter mentioned first, followed by the number. In the blank spreadsheet above, cell A1 is highlighted. You can move to other cells using the cursor keys ▼ ▲ ▶ ◀. Think of each cell as a box containing a number, its numerical *value*. When you start, all cells have no value, but you can change the contents of any cell by moving the cursor to the cell and entering a number, followed by =. To change the contents of a cell, move the cursor to the cell and enter a new value, which will replace any existing value. To clear the contents of a cell, highlight the cell with the cursor and tap DEL.

There are two items in the SET UP for spreadsheets, accessed from 4 *Spreadsheet* on the second page of SET UP and shown below. For now, set *Auto Calc* to be 1 On and *Show Cell* to be 1 Formula.

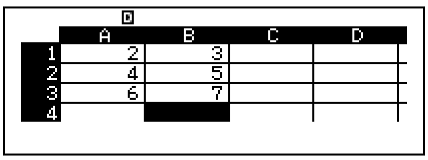






Basic calculations

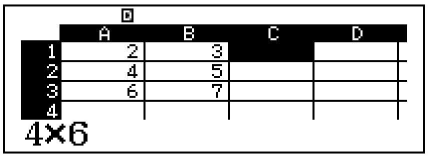
Spreadsheets are used to undertake calculations. To see how this is done, enter the values shown below into a new spreadsheet. Notice that when you tap =, the cursor moves down a row, so that it is easiest to enter data in columns.

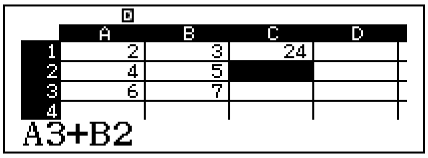


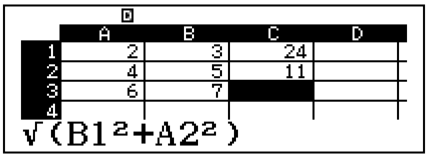
A cell value can be entered as a calculation directly. To see an example, move the cursor to C1 and enter 4 x 6, followed by =. If you make a mistake, you can edit the entry before tapping =. The result of 24 will be pasted into C1.

You can calculate a value of a cell from the value in other cells. For example, with the cursor in C2, enter A3 + B2, followed by [=], as shown in the second screen below. Use [ALPHA] [(−)] and [ALPHA] [°'"] respectively to obtain A and B, just as you do for memories. The result is that the values of A3 (6) and B2 (5) will be added to enter 11 in cell C2.

More complicated calculations can be made. For example, the third screen below shows an example, resulting in a value of 5 for cell C3.



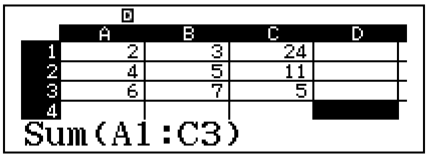


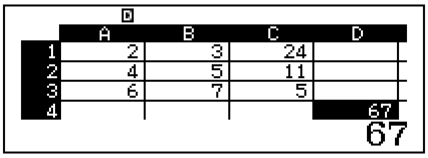


Move the cursor to D4 for the next step.

The *ClassWiz* also includes some useful calculations for a group of cells, as shown below. This menu appears as the fourth screen in the OPTN menu. Tap [OPTN] and then [▼] three times to access it. (In fact, you can also tap [OPTN] and then [▲] two times for the same result.)

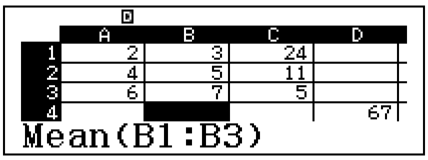
The names of these commands make their purpose clear. To specify the group involved, you need to indicate the first and last cell, separated by a colon ([ALPHA] [∫▫] on the keyboard). Thus, for example, the command Sum(A1:C3) will calculate the sum of all nine cells in the rectangular block with A1 in one corner and C3 in the diagonally opposite corner. In this case, that is all the nine cells. To enter this command in cell D4, once the cursor is in the cell, start with [4] Sum and then complete the command and tap [=], as shown below. (You will need to move the cursor around to see the screen exactly as below.)

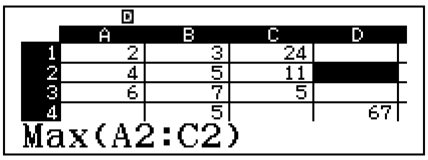


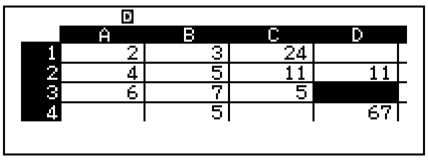


To find the mean or average value in the B column, enter the command Mean(B1:B3) in cell B4.

To find the maximum value in the second row, enter the command Max(A2:C2) in cell D2. These calculations are shown below.





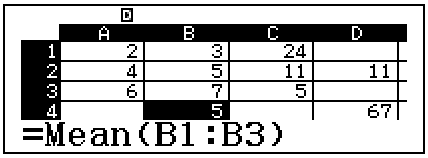


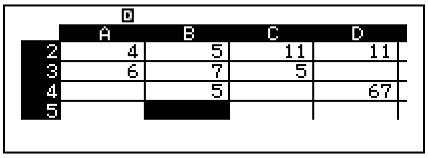
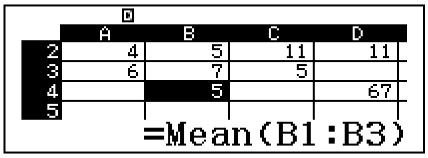
You can check for yourself that the calculations are correct.

Using a formula

The power of spreadsheets arises from the use of a *formula* for calculations, not just the calculations themselves. As in mathematics, a formula is an instruction to combine variables in a certain way. In the case of a spreadsheet, the contents of each cell is regarded as a variable.

A formula in the *ClassWiz* spreadsheet always starts with an equal sign, obtained by using ALPHA CALC, not the regular = key. To see how this works, enter a formula for the mean of the B column in B4, this time with an equals sign at the start, as shown in the first screen below.

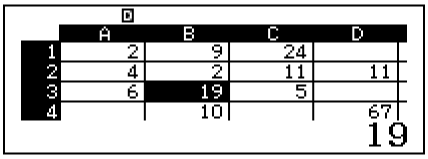


When you tap = to finish entering the formula, the previous cell value of 5 is replaced by the result of the formula, which is 5, so it appears in the middle screen as if there is no effect. Yet if you move the cursor to B4 again, notice that the result at the bottom of the screen shows that the cell contains the formula entered. The third screen shows this.

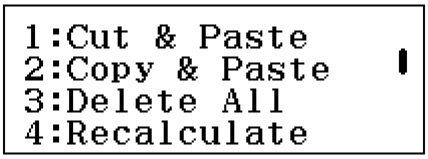
Cell B4 contains a *formula*, which has a *value*. The value is always shown in the spreadsheet itself. Because we SET UP the spreadsheet to show the formula in *Show Cell*, it is visible whenever the cursor is on the cell. (It is not visible, but is still there, if *Show Cell* is SET UP to show the value.)

Unlike the original calculation to calculate the mean, the formula will calculate the mean whenever any of the relevant spreadsheet values are changed. To see how this works, change the values in B1, B2 and B3 and observe that the mean is recalculated automatically each time a new value is entered. The example below shows this.



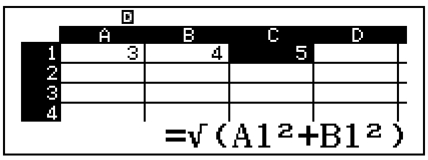
The mean in B4 is changed by the changes in B1, B2 and B3. Notice that the previous calculations in D2 and D4 are *not* changed (as they did not involve spreadsheet formulas).

If the mean value was not changed automatically on your spreadsheet you will need to check that the Auto Calculation is set to *On* in SET UP, as shown earlier. If it is set to *Off*, you can still recalculate values manually with 4 *Recalculate* in the second OPTN screen, as shown below.



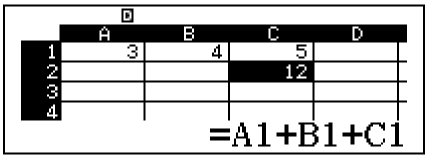
A formula is not restricted to the four special calculator operations in the OPTN menu. Any command involving the spreadsheet cells, and preceded by an = sign is a formula. Let's consider an easy example. Start by deleting all spreadsheet entries with OPTN ▼ 3 *Delete All*, as shown above.

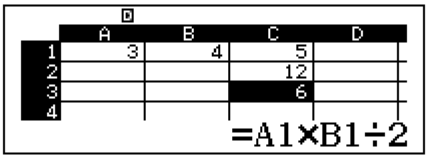
The spreadsheet below has the lengths of the two shorter sides of a right triangle in A1 and B1, and a formula in C2 to find the length of the hypotenuse of the triangle, using the Theorem of Pythagoras.

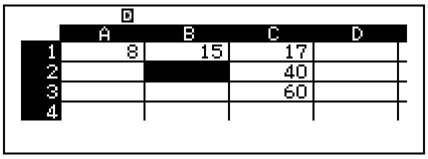


Construct this spreadsheet and try it out. Whenever you change values in A1 or B1, the new hypotenuse is calculated in C1. (This is similar to the CALC command described in Module 3.)

A spreadsheet may include several formulas, of course. For example, the above spreadsheet for right triangles might also include a formula to find the perimeter and the area of the triangle with separate formulas in C2 and C3 for this purpose, as shown below.





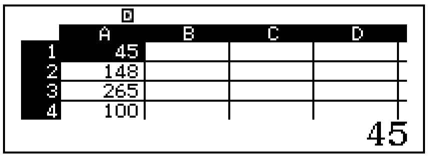


Try using this spreadsheet yourself. Any changes to the two side lengths in A1 or B1 will result in all three values in column C being recalculated immediately, as the third screen above illustrates.

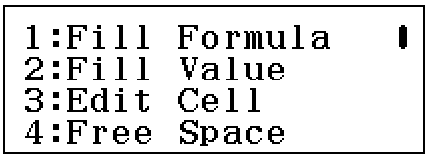
Fill commands

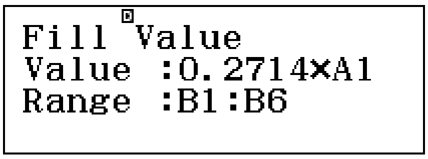
While formulas provide power and versatility to spreadsheets, their most powerful use involves dealing with a range of cells at once, including entering data automatically into new cells. A *fill* command is involved with this. There are two kinds of fill command available on *ClassWiz*, depending on whether you choose to put numerical values or formulas into a range of cells.

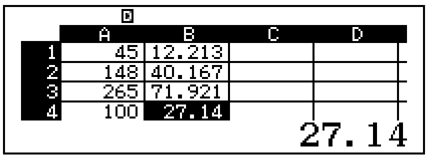
To see an example, we will make a spreadsheet to convert from one currency to another. Delete the contents of the spreadsheet to start a new spreadsheet, and store some decimal values (in your own currency) in the first six rows of column A. The spreadsheet below has some values in Malaysian ringgit.



To convert Malaysian ringgit to US dollars at present, it is necessary to multiply by 0.2714. So we need to multiply each of the values in the spreadsheet by 0.2714. The results will be stored alongside the values in the spreadsheet in Column B. A *Fill Value* command will do this with a single command, as shown below. With the cursor in cell B1, tap OPTN to access the spreadsheet options and then 2 *Fill Value* to access the command shown below. The command has two parts: the formula to be used and the range of cells to which it applies.



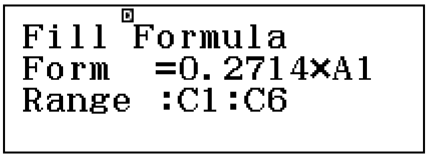


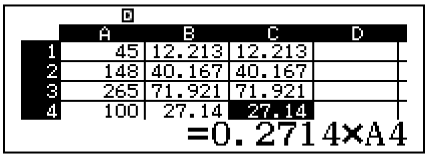


Enter the formula first, followed by =. Include in the formula the first cell in the range of cells (in this case A1 to A6). Use the ◀ and ▶ cursors and the DEL key to edit the range to finish at the final value (in this case, B6), followed by =. Tap = again to execute the Fill command, so that the formula will be applied immediately to *each* of the elements A1 to A6 and the results will be stored in B1 to B6, as shown above.

Check with the cursor that all six cells in the B column have been completed. The conversion from 100 ringgit to $27.14 is clearly correct, and a mental check suggests other values seem correct too.

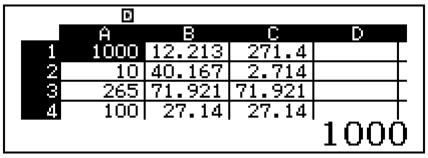
The other kind of fill command, accessed with OPTN 1 *Fill Formula* inserts a formula as well as a value in each cell. In other respects, it seems the same, as the screens below show, where the command has been entered in cell C1 and the results stored in C1 to C6. Notice, however that the formula line now includes an equals sign (automatically provided by the *ClassWiz*)





The values seem identical to those in column B. Check with the cursor that all six cells in the C column have been completed. Notice as you move the cursor down the column that *each cell now contains a different formula but the appropriate formula* for the line concerned.

As you saw in the previous section, a formula will be executed with any change of value. To see this, change some values in column A and notice that they are converted to new values for column C (but not for column B, which only contains values, not formulas). The screen below shows two examples in rows 1 and 2. The values in column B are unchanged by changing values in column A.



Note that you could have entered the six formulas one at a time in column C, using an equals sign for each, but it is much more efficient to use a *Fill Formula* command

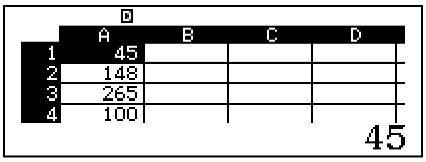
Absolute and relative addressing

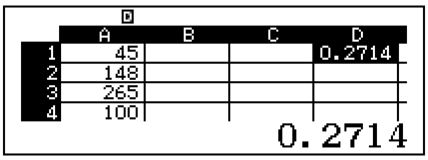
In the last example, the spreadsheet automatically generated the correct formula for each cell in the range. To do this, it compared the positions of the cell in the formula (A1) with the first cell in the range (B1): A1 is to the immediate left of B1. The same rule is applied to each row of the range in turn:

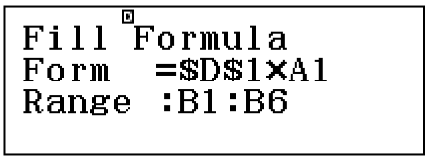
A2 is to the immediate left of B2
A3 is to the immediate left of B3
and so on …

This is called *relative addressing*, as the cell addresses are related to each other in the same way. Formulas usually work like that in a spreadsheet.

Sometimes it is necessary to also use *absolute addressing* in a formula, so that the same cell is referred to each time. Let's look at a different currency example, starting again with the original spreadsheet, shown at left below. Now add the conversion rate 0.2714 to cell D1, as shown in the middle screen.

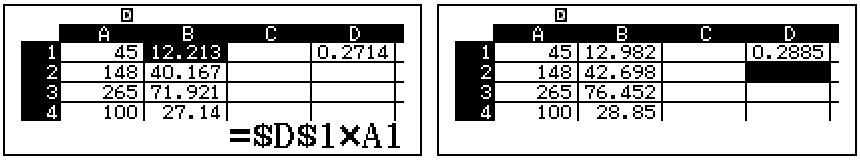






Finally, use Fill Formula command in cell B1 to do the conversions in column B. This time, instead of using the value 0.2714 in the formula, we will refer to cell D1 for each row. To show that the same cell is used for each row, the cell name is written with dollar signs as $D$1, as shown in the right screen above. The $ symbols are available with OPTN 1.

When you execute this formula, the result gives the same values as before, but the formula in each cell in the B column is different from before. Compare the first screen below with the earlier one:



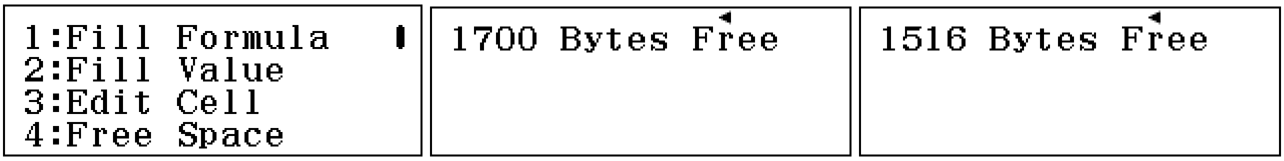
Each formula now uses both absolute addressing (e.g., $D$1) and relative addressing (e.g. A1)

But now, if the exchange rate changes to, say, 0.2885, then all that is needed is to change the value in cell D1, and all the conversions will be automatically made with the new value, as shown in the second screen above, because the formulas use the absolute address for D1. Check for yourself that the new conversions are correct. One way to do this is to change some values in the A column.

In this case, a much more efficient spreadsheet is available through absolute addressing.

Memory

The *ClassWiz* allocates only a certain amount of memory (1700 bytes) to the spreadsheet, and will not allow you to exceed that amount. Each value in a cell requires 10 bytes of memory, so a maximum of 170 (out of 225) cells can be filled with numbers. Formulas require extra memory, depending on their contents (note that a cell with a formula contains *both* the numerical value and the formula). The third screen below shows the present free space with the spreadsheet in use; in this case seven cells contain numbers at 10 bytes each, and six cells contain formulas and numbers at 19 bytes each, for a total of 184 bytes. (It is unnecessary for you to do these calculations.)



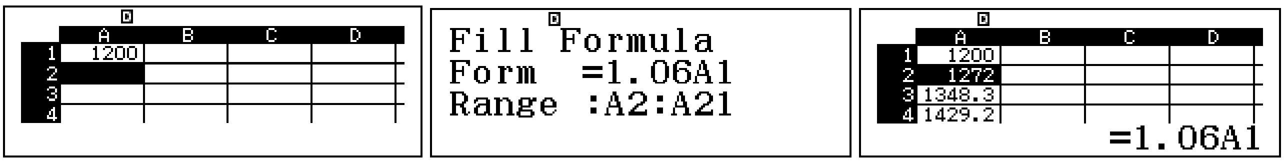
If a *Fill Value* command is used, it will consume less memory than a *Fill Formula* command, although it will usually be less powerful, as you have already seen.

We suggest that you check the free memory space (using OPTN 4) regularly, especially if you are constructing a large spreadsheet, in order to use the available memory wisely. For example, in some cases, the number of rows in the spreadsheet can be reduced without causing a problem and significant memory freed up accordingly. Tap OPTN or AC to return to the spreadsheet after checking.

Financial modelling

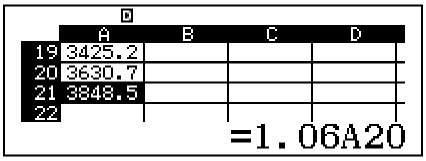
Spreadsheets are often used for financial modelling purposes, as they allow different financial arrangements to be studied.

For example, consider the case of compound interest. Suppose a sum of money, say $1200, is deposited into a bank account at 6% interest per annum, compounded annually. Start a new spreadsheet and enter the initial deposit in A1.

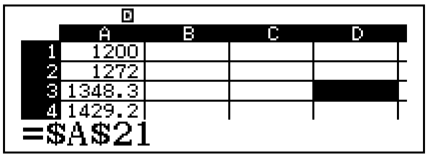


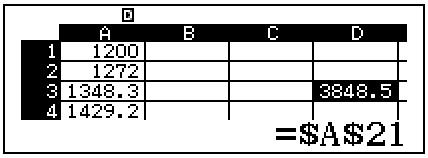
Following 6% interest, which is kept in the account, the amount in the account each year is increased by 6%. Use Fill Formula to store the balance every year for the next 20 years as shown above. Each row of the spreadsheet represents a year.

Scroll down to the bottom of the column entries to see that the deposit will grow to more than \$3848 in twenty years time (i.e., in the 21st year).

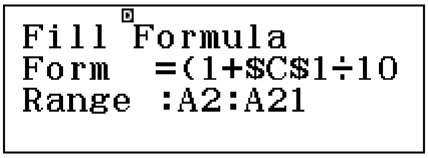


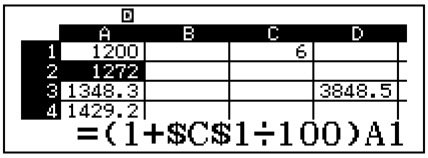
This spreadsheet can be improved in several ways. For example, to avoid needing to scroll down to cell A21 at the bottom, the result can be copied (using a formula with an absolute address) to the opening page of the spreadsheet, in cell D3, for example, as shown below. This improvement makes the spreadsheet easier to use.



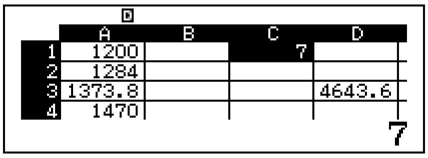


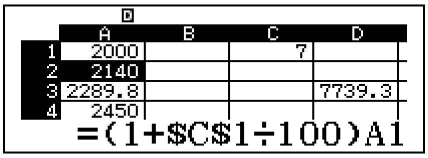
Another improvement is to make the spreadsheet more flexible. Instead of a compound interest rate of 6% per annum, a spreadsheet that allows you to model the effects of different interest rates can be obtained by making the interest rate a variable. In the version below, the interest rate (per cent) is in cell C1, and the fill formula adjusted to use that rate, using absolute addressing, as shown below. Notice that the entire formula is not visible in the first screen, but can be seen in the second screen.





Now to model the effects of a change in interest rate, from 6% to 7% per annum, all that is required is to change C1 to 7. The total after twenty years changes to more than \$4643:

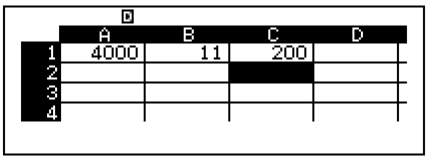




Similarly, to change the investment from \$1200 to \$2000, just change the amount in A1. The second screen above shows that the account will hold more than \$7739 after 20 years.

Similar spreadsheets allow modelling of other financial situations such as repayment of a loan. For example, suppose a loan is to be repaid with regular monthly payments and that there is an interest rate charged every month on the outstanding balance. How long will it take to repay the loan? What will be the effects of changing the repayments? What is the effect of choosing a loan with a different interest rate?

Start a new spreadsheet and enter the loan amount in A1. Enter the annual interest rate (to be interpreted as per cent) in B1 and the proposed monthly repayment in C1. The screen below shows a car loan of \$4000, with an interest rate of 11% p.a., and monthly payments of \$200.



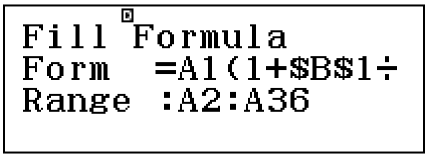
In column A, we will use *Fill Formula* to show the balance at the start of every month. The key idea is that each month

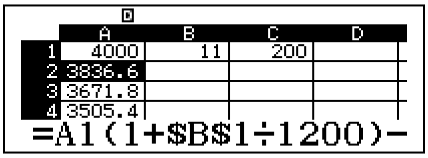
new balance = previous balance + interest – repayment

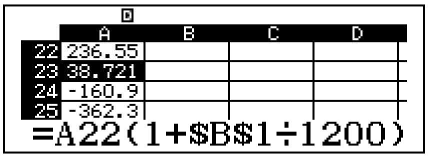
A suitable Fill Formula for this, in cell A2, is

=A1(1+$B$1÷1200)-$C$1

The monthly interest is one twelfth of the annual interest. This formula is not shown in full in the screen below, as it is too long for the display, but you will be able to enter it successfully. The formula generate the balances at the start of each month for 36 months or three years:

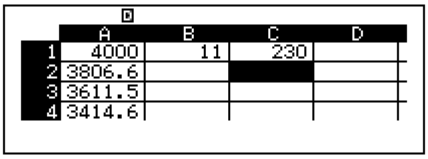


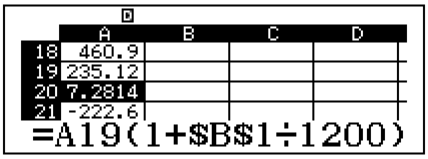




Scrolling to the bottom of the list, shows that the balance is $38.72 at the start of the 23rd month, so that a final payment of less than $200 should then be made, and it is not necessary to continue making payments for three years. Note that you can scroll up with ⏶ as well as down with ⏷.

If a larger payment of $230 instead of $200 were to be made each month, the loan can be repaid more quickly, in just over nineteen months, as shown below.





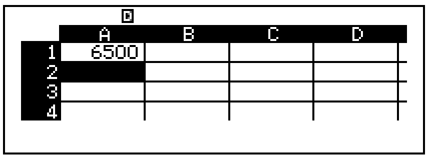
You can also explore the effects of interest rate changes by adjusting B1 or different loan amounts by changing A1.

Sequences and series

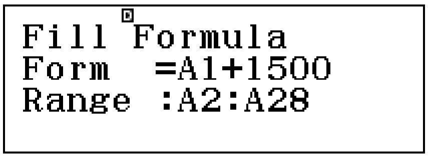
A spreadsheet is an ideal tool for modelling and studying sequences and series, as successive terms can be generated and added. These topics are also addressed in Module 14, using other *ClassWiz* capabilities.

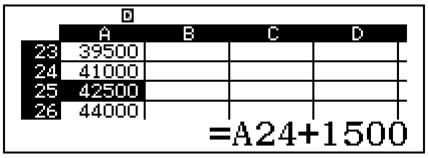
For example, suppose an athlete is training for the Marathon, a race of 42.195 km. She decides to increase her endurance by running 6500 m on the first day, 8000 m on the second day, 9500 m on the third day, and so on, increasing her distance by 1500 m each day. When will she first run a Marathon distance? How far will she have run altogether by that time?

Start a new spreadsheet, with the first distance in A1.



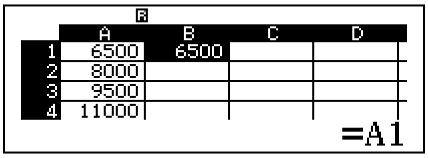
Use a Fill Formula in A2 to find the distance run on later days for a four-week period, as shown below. The values in column A are described as a *sequence*.



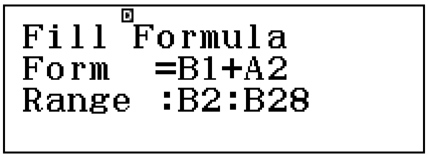


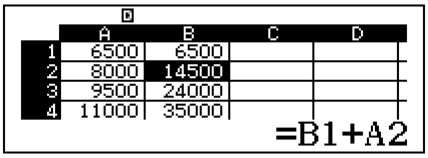
The spreadsheet shows the sequences of distances to be run each day, as each day is represented by a row. Scrolling down the spreadsheet reveals that the first day of her proposed training schedule in which she runs the full Marathon distance is Day 25.

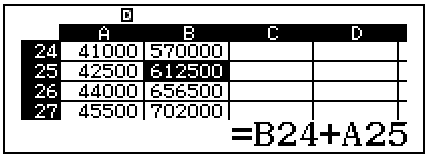
To see the total distances run by the end of each day, it is necessary to add successive distances. We will record the totals in column B, after using a formula to copy the first value into B1:



Each day, the previous total is increased by the distance run on that day, as shown in the A column. So, a suitable Fill Formula in B2 to calculate the progressive total each day is shown below. Check mentally that the first few values showing are correct. These successive values comprise a *series*.







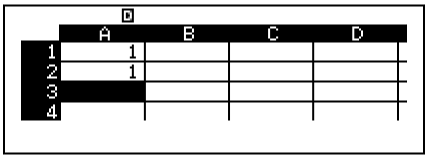
Scrolling down column B shows that, by Day 25, she has run a total distance of 612.5 km.

The Fibonacci sequence

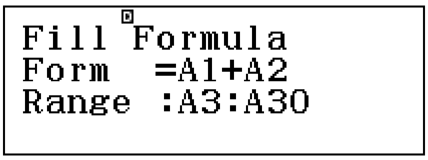
A famous sequence has been used to model a population of rabbits that starts with one pair. In their first month, they do not breed, but in every month after that, they produce a pair of children. The children likewise do not breed in the first month, but do so every month afterwards, and so on. Furthermore, the rabbits do not die. This sequence is called the *Fibonacci sequence*, named after a medieval Italian mathematician. Check for yourself that the number of pairs of rabbits each month is then:

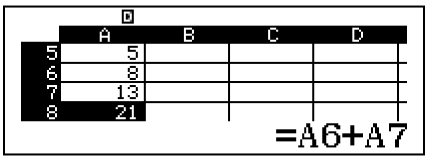
$$1, 1, 2, 3, 5, 8, \ldots$$

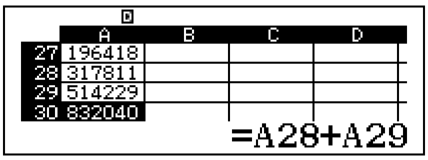
To model the rabbit population growth, start a new spreadsheet, with the population (in pairs) at the start of the first two months in A1 and A2, as below.



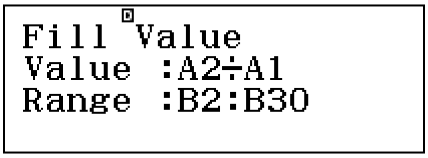
Each successive term in the A column is the sum of the previous two terms, so a suitable fill formula for generating the first thirty of these is shown below, together with some terms. Check for yourself that these are correct.

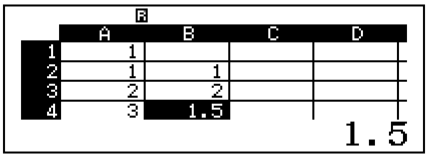




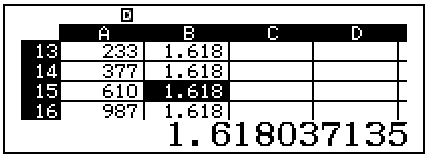


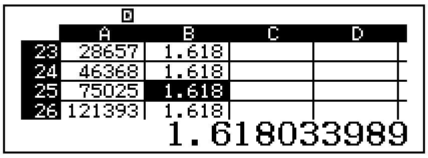
An interesting mathematical question concerns the ratio of the number of rabbits in successive days. Thus, after Day 3, the ratio is 2:1 =2, after Day 4 it is 3:2 = 1.5, after day 5 it is 5:3 = 1.667, etc. To study how these ratios change over time, a Fill Value command is helpful, as shown below, starting in cell B2. (Fill Value is better than Fill Formula here, in order to see more decimal places for the results.)

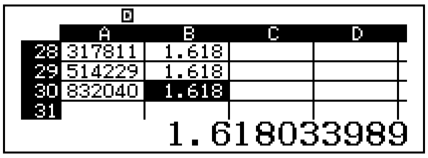




If you examine these ratios in column B, you will notice the startling fact that they get closer and closer to a single value, the famous *Golden Ratio* $\Phi \approx 1.61803\ldots$, often associated with aesthetics.







The ratios are not exactly the same each time, even though they seem to be so on the last two screens above, but the spreadsheet helps you to see that they *converge* to a single value (which in fact is an irrational number, so cannot be a ratio of two integers).

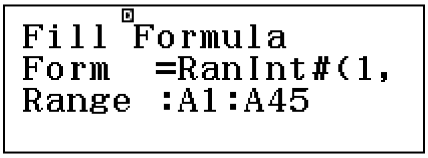
Modelling randomness

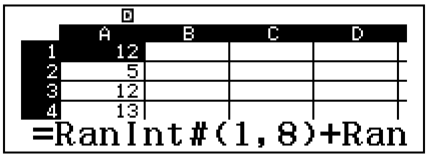
In Module 12, you saw that random events can be simulated with the *ClassWiz*. Using a spreadsheet allows you to gather many random observations quickly to see their results and any trends.

For example, consider tossing a pair of 8-sided dice, each of the faces of which are equally likely to occur. What is the likely result of adding numbers showing on the two faces? To model the dice sums, we need a formula that is too long to show completely on the *ClassWiz* screen, but which can be entered without difficulty, using the random integer command, with ALPHA •:

=RanInt(1,8)+RanInt(1,8).

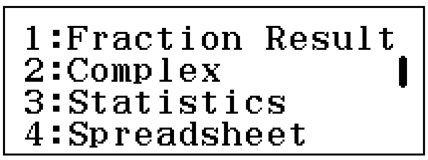
As the screen below shows in a new spreadsheet, 45 pairs of dice rolls (the maximum number available on *ClassWiz*) have been simulated in this case.

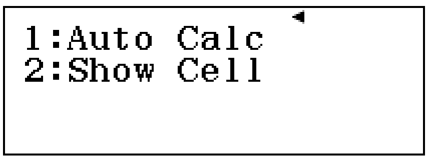


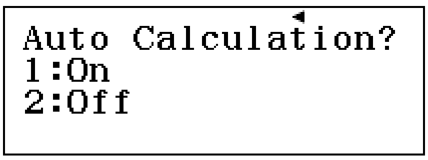


Because the data are random, it is (very) unlikely that your results are the same as those shown above. If you scroll the data, you will get a sense of the typical values obtained. Compare your observations with others as well.

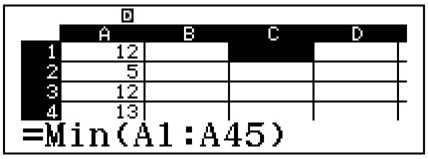
A more efficient study of the data, however, uses the spreadsheet to do some of the analysis. First, temporarily turn off the Auto Calc in SET UP, as shown below:

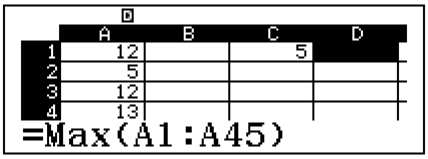


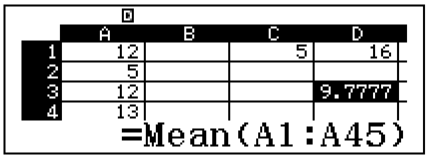




Then insert the formulas to find the minimum value, maximum value and mean of the values in C1, D1 and D3 respectively as shown below. Don't forget the equal signs.



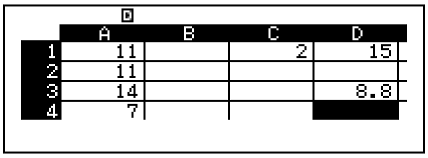


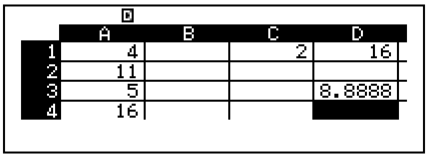


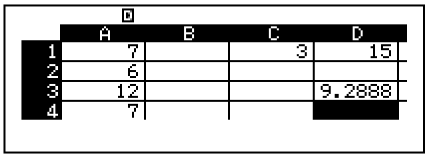
These screens show that our first set of 45 rolls has a mimimum sum of 5 (which exceeds the possible minimum of 2), a maximum of 16 (which is the maximum possible) and a mean sum of 9.78. Your results are unlikely to be the same as these in all respects.

Now turn the Auto Calculation back on in SET UP.

Once the spreadsheet is set up in this way, you can study 45 dice rolls by recalculating the spreadsheet (Tap OPTN ▼ 4 to do this), which has the effect of generating a fresh set of rolls. For example, here are three successive sets of 45 rolls we obtained, together with their statistics:



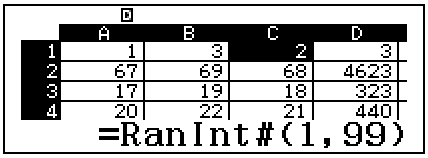


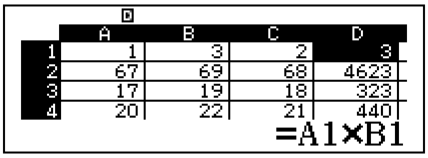


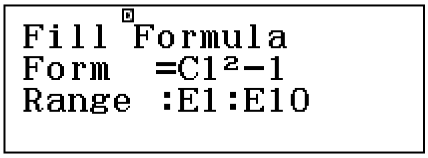
Although different results are obtained each time, there are similarities too. Compare your results with those of others. It is likely that the mean scores are close to 9 each time, the minimum is likely to be 2 or 3 and the maximum is likely to be 15 or 16. Of course, as this is a random experiment, these will not occur *every* time (and did not occur in our original attempt above.)

Modelling variables

As a final example, consider the algebraic expression $(x + 1)(x - 1)$ obtained by multiplying $(x + 1)$ and $(x - 1)$. A spreadsheet can be used to see many examples of this, In the new spreadsheet below, ten random integers (i.e., values for x) are stored in column C, $(x - 1)$ is stored in column A, $(x + 1)$ is stored in column B, and the product $(x + 1)(x - 1)$ is stored in column D using fill formulas.

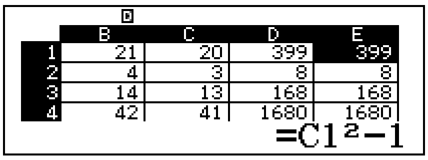


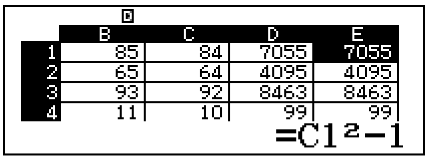


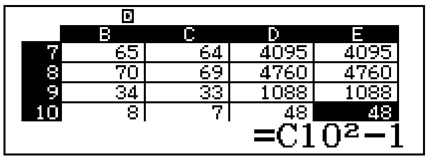


Each time the spreadsheet is recalculated, a fresh set of values for x and so for $(x + 1)(x - 1)$ is obtained. Now enter one more fill formula into column E to calculate $x^2 - 1$, as shown above.

As shown below, each time the spreadsheet is recalculated, and so different values are obtained for x (in column C), the values in column D, representing $(x + 1)(x - 1)$, and the values in column E, representing $x^2 - 1$ are *identical*. You can check further values each time by scrolling down, since the spreadsheet only shows four rows on each screen.







The spreadsheet helps to show the meaning of the *identity* $(x + 1)(x - 1) = x^2 - 1$, which is *always* true regardless of the value of x.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. Enter the following data into a new spreadsheet

	A	B	C	D
1	4	6	11	
2	2	8	2	
3	9	13	5	
4				

   (a) Enter a command (not a formula) into cell D1 to add A1 and B2.
   (b) Enter a formula in cell D2 to evaluate A2+B2+C2; then check by changing A2 to 7.
   (c) Enter a formula in cell D3 to find the mean of all the six cells in columns A and B; then check by changing B2 to 3.
   (d) Enter a formula in cell D4 to find the maximum value of the nine cells in columns A, B and C; then check by changing B1 to 37.

2. Use a *Fill Value* formula in a new spreadsheet to count by tens in column B, starting with 3 in cell B1 and 13 in cell B2. Then check that changing B1 does not affect the rest of the column.

3. (a) Enter 5 in cell A1 and 7 in cell D1 in a new spreadsheet.
   (b) Use a *Fill Formula* with absolute addressing to count by 7's in column A down to cell A30.
   (c) Change D1 to 5 and check that column A then shows counting by 5's down to 150.

4. (a) In a new spreadsheet, use a formula to create a sequence in column A that starts with 1/6 and for which each successive term is 1/6 more than the previous one. (Use the fraction key to do this, even though the *ClassWiz* automatically shows the fractions in Line mode.)
   (b) Use SET UP to change Show Cell to *Value* instead of *Formula*, and check that the results are shown as fractions at the bottom of the screen and decimals in the spreadsheet.
   (c) What is the final value in the column, A45?

5. Make a new spreadsheet to produce a table of values of the function $f(x) = x^2 + x$. Start with several values for x in column A and then use a Fill Formula to generate values for the function in column B. Check mentally that the values are correct.

6. (a) Make a new spreadsheet to produce a table of values of a function. Start by using a Fill Value command to enter the counting numbers from 1 to 45 in column A.
   (b) Enter 2 into Cell D1.
   (c) Use a Fill Formula in column B so that each value in column B is the logarithm of the value to the left in column A, with the base of the logarithm in D1. Use the key but note that in Spreadsheet mode, the command is written automatically in Line mode as log(a,b).
   (d) Check that your spreadsheet shows $\log_2 32 = 5$.
   (e) Change D1 to 5 and check that $\log_5 25 = 2$.

7. Consider carefully the following spreadsheet and the Fill Formula command shown in cell A4.

	A	B	C	D
1	4	6	11	
2	2	8	2	
3	9	13	5	
4				

```
Fill Formula
Form  =Sum(A1:A3)
Range :A4:C4
```

   (a) Describe the effect of entering the Fill Formula. Check by doing this on your *ClassWiz*.
   (b) What will be the effect of changing the Fill Formula to =Mean(A1:A3)? Check by doing this on your *ClassWiz* and also by using SET UP to change *Show Cell* to *Formula*.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. Design and build a spreadsheet for repaying a loan of $10 000 using monthly payments when there is an interest rate of 9% per annum, compounding monthly. Use your spreadsheet to find the monthly payments needed to repay the loan in 24 months, and also calculate the total amount of money repaid in that time. Ensure that your spreadsheet is flexible, so that the original loan and the monthly repayments can be changed without requiring a new spreadsheet to be constructed. Investigate the effects of a change in interest rate.

2. A radioactive substance is decaying at a steady rate, and each year loses 2% of its mass. (So, after one year, it will have only 98% of the mass it started with.) Construct a spreadsheet to model this situation.

   (a) How much of the substance will remain after 10 years?

   (b) After how many years will the substance be half gone? (This is called the *half-life*.)

   (c) Predict what the half-life would be for a different rate of decay, such as 4%; check your prediction with your spreadsheet.

   (d) Investigate other rates of decay.

3. Use a spreadsheet to examine sequences like the Fibonacci Sequence for which each term is the sum of the previous two terms. For example, 2, 5, 7, 12, 19, 31, … Try several different examples, and compare the ratio of successive terms as we did with the Fibonacci Sequence in the Module text.

4. (a) Make a new spreadsheet to investigate what happens if a pair of fair 12-sided dice is tossed and their scores added. Include a facility in your spreadsheet to find the mean of the totals. Use your spreadsheet several times to see how much the means vary.

   (b) Compare your results with those of another person.

   (c) Make a new spreadsheet to study what happens if three fair 6-sided dice are rolled and their scores added. Compare your results with others.

5. (Sequences are treated in Module 14, so you may like to try this activity later.).
   Investigate what happens eventually with the recursive sequence given by $T_{n+1} = kT_n^2 - 1$ for different values of k, with $1 \le k \le 2$.

   (a) Use a spreadsheet to start with $T_0 = 0.4$ and with $k = 1$. (Check that the first few terms are 0.4, -0.84, -0.2944, -0.9133, …)

   (b) Try some other starting values for T_0, with $0 \le T_0 \le 1$.

   (c) Once you have found what happens with $k = 1$, investigate larger values of k, such as $k = 1.1$ or $k = 1.4$. Compare your results with others.

   (d) With a fixed value of k, investigate the effects of small changes in the starting value T_0.

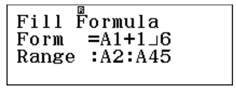
6. Use a spreadsheet to investigate $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x...}}}}}$ for various integer values of x. Start with $x = 2$ and then try $x = 1$ and other values.

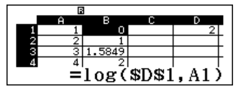
Notes for teachers

This module highlights the ways in which the *ClassWiz* can support students to learn how to use a spreadsheet for some mathematical purposes. The text of the module is intended to be read by students and will help them to see how the calculator can be used to design and operate spreadsheets, especially those that involve formulas. Some examples of situations that can be effectively modelled with spreadsheets are briefly treated. Some work in the following Modules make use of spreadsheets. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. (a) A1+B2 will result in 12 in D1 (b) Use =A2+B2+C2 or =Sum(A2:C2) to get 12; changed value is 17 (c) =Mean(A1:B3) will result in 7.83; the change results in 9 (d) =Max(A1:C3) results in 13; changed value is 37 2. Formula in B2 is B1+10, Range: B2:B45 3. Formula in A2 is =A1+$D$1, Range: A2:A30 4. (a) see the formula in A2 at right (c) 7.55. Formula in A2 is A1+1, formula in B1 is =A1^2+A1 6. (a) A1=1, formula in A2 is A1+1, Range:A2:A45 (c) see the formula in B1 at right 7. (a) column sums are calculated: A4=15, B4=27, C4=18 (b) Column means are calculated: A4=5, B4=9, C4=6.

4. (a)


6. (c)


Activities

1. The *Financial Modelling* section of the text gives an example of how students might set up their spreadsheets. They should try various monthly payments amounts and check which gives close to a zero payment in the 25th month. Encourage students to include a formula in their spreadsheets to calculate the total payments. [Answers: $440.17 per month gives total payment of $11 004.]

2. Radioactive decay is well modelled with a recursive sequence (see Module 14). The mass each successive year is 0.98 times the previous mass, so a formula in A2 like =0.98A1 is appropriate. Students can scroll down to check when mass is reduced to 50% of original. Encourage students to explore changes to the rate of decay. [Answers: (a) 81.7% (b) about 34 years (c) about 17 years]

3. Spreadsheets like those for the *Fibonacci Sequence* in the text are appropriate. Encourage students to use absolute addressing for the first two terms for easy explorations. Very surprisingly, the ratios of successive terms converge to the Golden Ratio, regardless of starting values.

4. Spreadsheets like those used in *Modelling Randomness* in the text are appropriate, with the formula changed for suitable dice. Encourage students to share results in class, so that the small variations of the means are seen as well as the consistency of the random results. [Answers: (a) means will be close to theoretical mean of 13 (c) means will be close to theoretical mean of 10.5]

5. The recursive sequence here will result in *chaos* for larger values of *k*, and students might be encouraged to search for discussions and applications of this idea on the web. A spreadsheet that uses absolute addressing for the variable *k* will make explorations easier. Encourage students to calculate 45 terms of sequences to study effects. [Answers: (a) sequence eventually alternates with -1, 0, -1, 0, … (b) there is no effect, surprisingly (c) low *k* values lead to patterns late in the sequence, but $k > 1.4$ leads to chaotic sequences (d) when sequences are chaotic, small changes to the starting value have dramatic and apparently random effects on the sequence (d) for $k > 1.4$, sequences exhibit 'sensitive dependence on initial conditions', characteristic of chaos.]

6. Students will probably need help and a class discussion to set up a suitable spreadsheet, such as A1 = 2, A2 formula =√(2+A1). [Answers: $x = 2$ converges to 2, but $x = 1$ converges to the Golden Ratio, very surprisingly. To see why, solve the equations $x = \sqrt{(2+x)}$ and $x = \sqrt{(1+x)}$.]

Module 14
Recursion, sequences and series

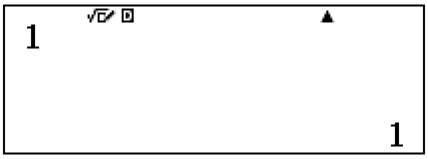
Sequences and series are important components of mathematics, each of which arises from the mathematical idea of recursion. The *ClassWiz* offers a number of different ways of dealing with these mathematical ideas, using Calculator, Table and Spreadsheet modes.

For this module, make sure your calculator is set into Math mode for both input and output and *Norm 2*. Use SET UP ([SHIFT] [MENU]) and then [1] or [3] to do this, if necessary.

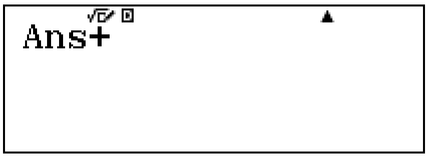
Recursion

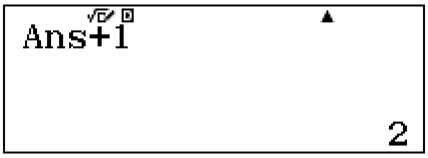
A basic idea of recursion is to use the same process repeatedly in order to produce a succession of numbers. Perhaps the most fundamental example in mathematics involves counting, where the process involves adding one more to a number in order to reach the next number. The recursive process of adding one more can be carried out automatically by the calculator. To see how this works, we will study one example in detail.

Start in Calculate mode with entering the first number and tap [=]. In the example below, we have started with 1, although of course you can start counting at a different number if you wish.

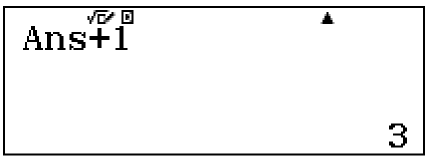


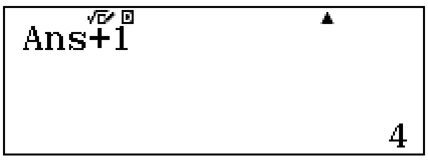
In this case, the recursive process is to "add one", so enter [+] [1] on the calculator. Because addition is a binary operation – that is, two things must be added – the calculator interprets a command to add one as a command to add one *to the previous result*. As soon as you tap the [+] key, the screen shows *Ans*+. The calculator memory *Ans* refers to the previous answer. Completing the calculation with [1] [=] produces 1 + 1 =2, as expected.

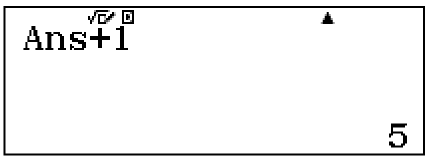




While it is of course generally unnecessary to use a powerful calculator to add one and one, the calculator has now been given a command to add 1 to the previous result. If you tap the [=] key again (with no command entered), the calculator will execute the most recent command again. In this case, it will use the command *Ans+1* to add one to the previous result every time you tap [=]. The next three screens have been obtained by tapping [=] three times in succession.





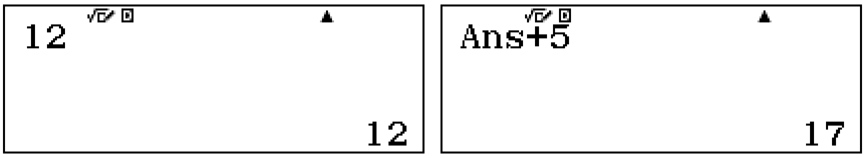


As you look at the screen, it seems as if nothing is changing each time you tap [=], except the result. The command *Ans+1* appears to be the same each time. However, if you check with the [▲] key, you will see that the command is actually *repeated* each time; it merely looks to be the same because it is the same command.

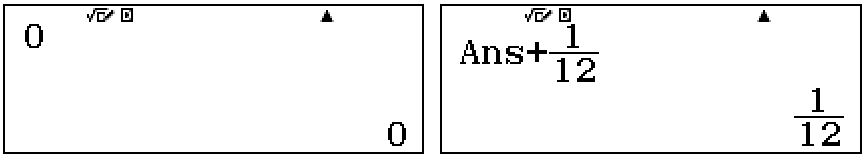
Clever counting

The recursive process described above using addition can be employed for other counting as well as counting by ones. You can start at different starting points and count by different amounts, by adjusting the first step and the recursive step appropriately.

For example, to count by fives, starting with twelve, the steps involved are shown below. Every time you tap the = key, another five will be added. It is possible to do this many times in succession.

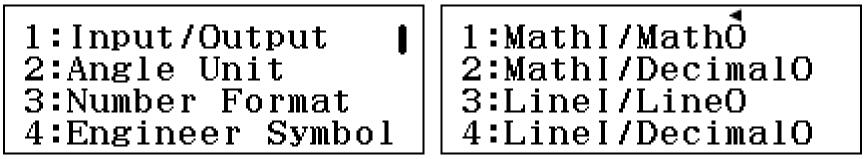


You can count by numbers that are not whole numbers, of course. In the example below, the calculator will count by one twelfths, starting with 0.

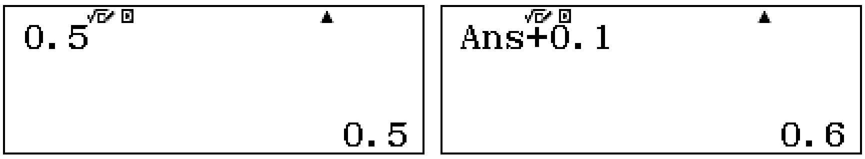


When you count in this way, you may be surprised at the results each time you tap the = key, as the calculator will automatically simplify the fractions concerned.

Sometimes, it may be unhelpful to have fractions displayed automatically. You can avoid this by tapping SHIFT = each time, but that is quite tedious. An alternative is to adjust the output to give decimal results temporarily, using SET UP (SHIFT MENU). After selecting *1:Input/Output*, select *2:MathI/DecimalO* to get decimal results output, as the screens below indicate.

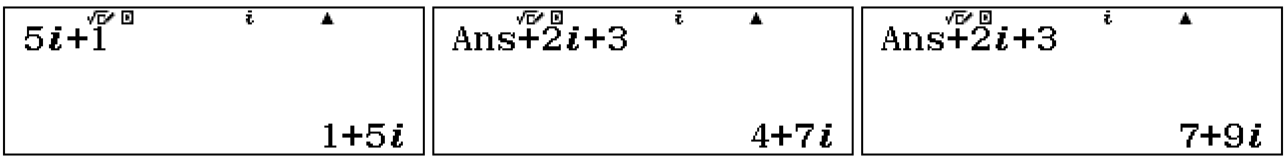


Then, counting by 0.1 from a starting point of 0.5, as shown below, will be easy to do with a tap of the = key for each term:



You may wish to change the calculator back to output results in *Math* mode, instead of decimals, depending on what you are doing next.

Recursive procedures can be used in other modes. For example, when the calculator is in Complex mode (as described in Module 9), it can be used to count with complex numbers. The example below shows counting by ($2\boldsymbol{i} + 3$) from a starting position of $5\boldsymbol{i} + 1$:

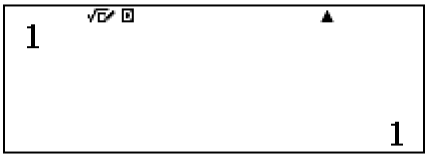


Notice that the real component of the numbers is increased by 3 and the complex component increased by 2***i*** each time you tap the = key.

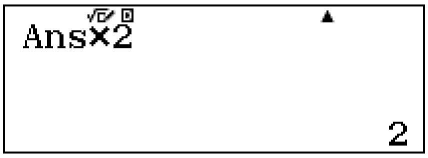
Recursion and multiplication

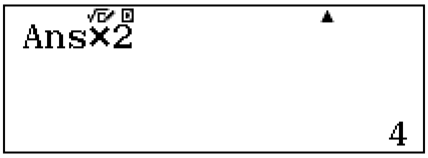
Using recursive procedures with addition results in counting of various kinds, as the previous two sections have shown. Recursion is also possible with other operations, including the important case of multiplication. When applied repeatedly, multiplication results in exponential growth, which is very important in the natural world and the world of finance.

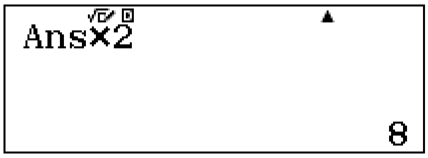
To see an example, start with 1 in the calculator:



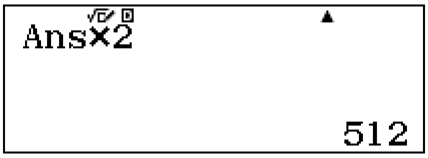
Then introduce the recursive step of multiplying by two. Each time you tap =, the calculator multiplies the previous result by two, which produces in turn successive powers of 2, as shown below.





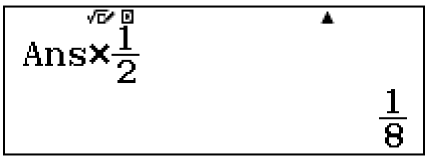


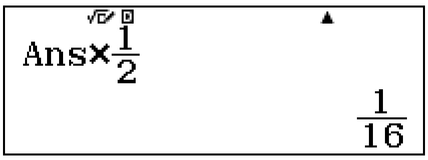
Check for yourself that, after tapping the = key nine times altogether, you will see $2^9 = 512$.

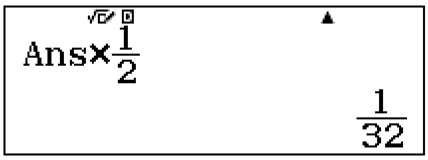


Make sure you count the number of taps carefully as you proceed.

If you start with 1 and multiply it by ½ successively, you will see the powers of ½ , these are the negative powers of two. [For example, ½ to the power 2 is the same as 2^{-2}.] The third, fourth and fifth terms are shown below:





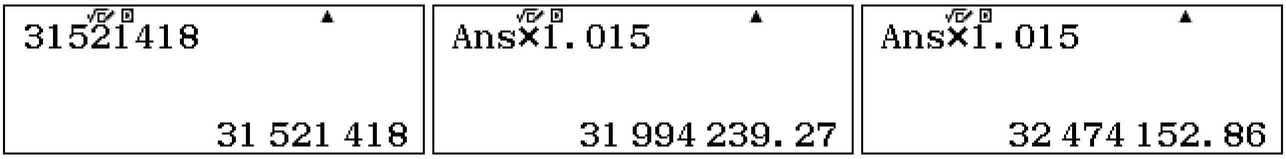


You can perhaps recognise these three terms are 2^{-3}, 2^{-4} and 2^{-5} respectively.

An important kind of multiplicative recursion involves growth processes in which each successive term is a certain multiple of the previous term. A good example is population growth, since it is common practice to describe the annual growth rate of a nation's population as a percentage of the size of the population.

The population growth rate of Saudi Arabia was estimated on the Internet in July 2015 as close to 1.5%, with an estimated total population of 31 521 418 people. If this growth rate continues, the population each year will be 1.5% higher than the previous year. The easiest way to obtain a number 1.5% higher than another number is to multiply by 1.015.

If this is done repeatedly on the calculator, the annual population can be predicted efficiently, as shown below.

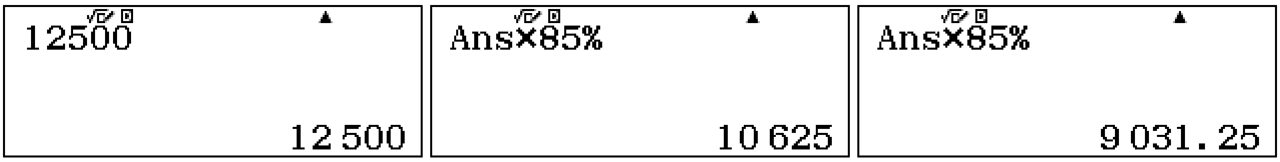


The other two screens suggest that Saudi Arabia's population will be about 31 994 239 in mid-2016 and about 32 474 152 in mid-2017.

You need to be careful interpreting these screens. In the first place, the original data are estimates. Secondly, the recursive procedure is based on an assumption that the population growth rate remains constant, although of course it might fluctuate in practice. Thirdly, the results are given to many decimal places, although it does not make sense to have a population that is not a whole number. With these cautions in mind, however, you should be able to check with your calculator that the population of Saudi Arabia is projected to reach 40 million people around 2031.

Numbers can become successively smaller with multiplication, provided the multiplication is by a number less than one. A process which decreases a quantity multiplicatively by the same amount each period is usually called *exponential decay*, as you saw in Module 6. A good example involves radioactive decay of materials, used in carbon dating. Another good example, in the man-made world, involves depreciation of values.

To illustrate depreciation as a recursive process, suppose that, for insurance and taxation purposes, a small business regards its office furnishings as depreciating in value by 15% every year. If it begins with furnishings valued at $12 500, you can use the calculator to construct a recursion to show the depreciated value of the furnishings every year. For this example, the multiplication each time is 85%, which is the proportion of the value that remains after each year elapses. The screens below show the effect of this depreciation process.(The percentage symbol is accessible with SHIFT Ans.)



Notice that it would have also been acceptable to use 0.85 instead of 85%, to indicate what value remains after each year. If you use this process carefully, and count how many times you tap the = key, you should be able to see for yourself that the furnishings have depreciated in value after seven years to only a little more than $4000.

Sequences

A *sequence* is a collection of numbers in a particular order. In mathematics, sequences usually have a well-defined rule for their construction. In general, sequences are written with each of the terms in order, separated by commas. You have seen some examples of sequences already in this module, including a sequence of counting numbers:

$$1, 2, 3, 4, 5, \ldots$$

and a sequence of powers of two:

$$1, 2, 4, 8, 16, 32, \ldots \text{ (i.e., } 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots)$$

Sequences can generally be defined in two ways, either *recursively* or *explicitly*. A recursive definition involves giving the first term and describing how the other terms are to be found, as the earlier sections of this module have demonstrated.

An explicit definition for a sequence involves giving a rule to find any particular term. Since the terms are described as the first, second, third, fourth, etc. … it is clear that only whole numbers can be used for terms, so it is common to express the nth term (often represented as $T(n)$ or T_n) as a function of the term number, n, for n a counting number.

Thus, in the first case above, the sequence is defined as:

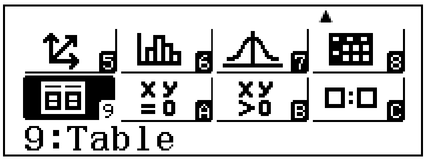
$$T(n) = n, \text{ for } n \text{ a counting number.}$$

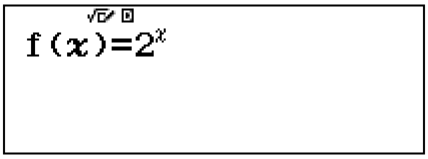
The powers of two can also be defined as a sequence:

$$T_n = 2^n, \text{ for } n = 1, 2, 3, 4, \ldots$$

While it is convenient to generate terms of a sequence recursively, as we have done earlier in this module, it is sometimes more efficient to generate a particular term, using an explicit definition. A table of values is helpful to generate terms of a sequence on the calculator, since we can think of a sequence as a particular kind of function (for which the variable can only be a whole number).

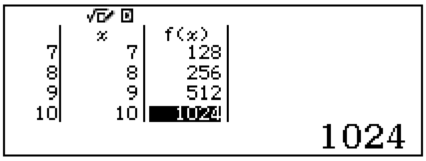
Set your calculator in Table mode, and define the sequence as shown below.

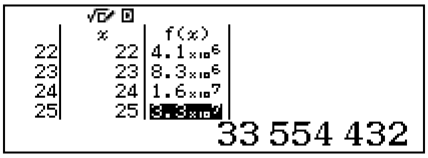




Notice that the calculator requires the use of x rather than n to represent the term number, and $f(x)$ to represent the term T_n, but the meaning of $f(x) = 2^x$ is the same as $T_n = 2^n$. Tap = to continue defining the sequence. The calculator is restricted to 30 terms of a sequence at any time. So the first 30 terms can be generated by the parameters shown above. Tap = after each entry.

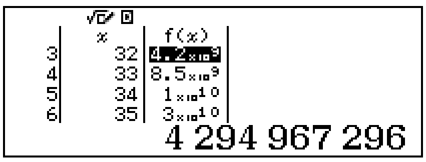
You can scroll up and down to find various terms of this sequence. The two screens below show the 10th term and the 25th term.

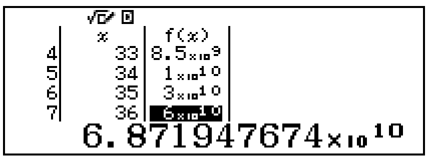




Notice that the 25th term is shown at the bottom of the screen (as the cursor is in the *f(x)* column adjacent to $x = 25$). The term is also shown in the table itself, but the small amount of space means that the calculator has displayed it in scientific notation.

If you wanted to find other terms of the sequence (such as the 32nd and the 36th term), you could change the table specifications. Start by tapping AC. Change the *Start* and *End* parameters, but leave the *Step* at 1.

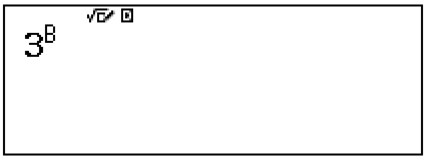




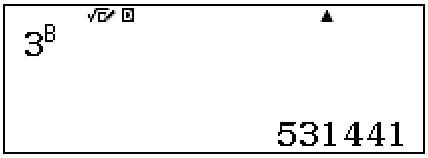
As the screens above show, the 32nd term is 4 294 967 296. Notice that it is next to $x = 32$. In this case, however, the 36th term cannot be displayed by the calculator in full, but can only be approximated by scientific notation, because it has more digits than the calculator screen can handle.

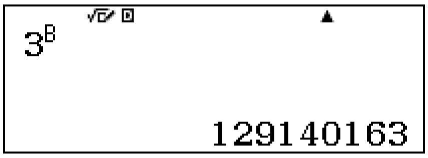
When you have an expression for the nth term of a sequence, it is also possible to evaluate a particular term, using the CALC facility of the calculator in Calculate mode. Sometimes, it is easier to use this than to construct a table of values, especially if only one or two values are required and you are less interested in the whole sequence of values.

To illustrate, consider the sequence consisting of powers of 3, given by $T_n = 3^n$, for $n = 1, 2, 3, \ldots$ Enter a suitable expression in the calculator; any of the calculator variables can be used for this purpose. To illustrate, we have used B for the example below.



Then tap CALC, followed by the desired value and = key. Do not tap AC (since it will clear the expression), but tap = again for the next desired value.





The two screens above show this process to find the 12th term and the 17th term respectively.

Arithmetic sequences

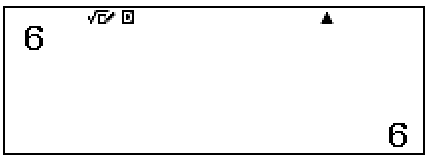
A particular kind of sequence that is important in mathematics is an *arithmetic sequence*, sometimes also called an *arithmetic progression*. This is a sequence for which there is a first term (represented by a) and a common difference (represented by d) between successive terms. That is, each term can be obtained by adding a constant number to the previous term.

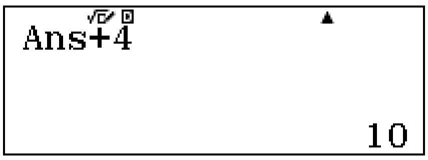
A good example involves taxi fares, when there is a set charge for each trip and a certain amount to be paid for each completed kilometre travelled. Suppose that $a = 6$ and $d = 4$. So the taxi costs $6 to start and $4 for every kilometre travelled. [In practice, many taxis may charge rates for parts of a kilometre and even other charges, but we will consider only a simple case here, for purposes of illustration.]

The sequence of costs for various kilometres travelled is an arithmetic sequence:

$$6, 10, 14, 18, 22, \ldots$$

This can be represented in the calculator either recursively or explicitly. The following screens show the recursive case. After a trip of 1 km, the fare is $10. (This is the second term of the sequence.)





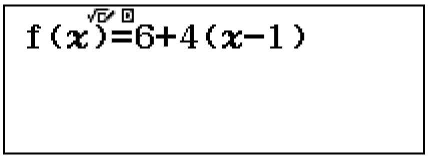
Any taxi fare can be determined this way, by tapping the = key the appropriate number of times.

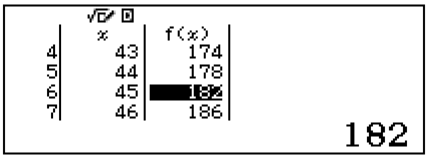
While this is perfectly acceptable for a short trip, it would be very tedious for a long trip (such as one of 44 km), so an explicit version of the arithmetic sequence may be better. In the general case, the *n*th term of an arithmetic sequence is given by

$$T_n = a + (n - 1)d$$

Since the nth term involves adding d to a a total of $n - 1$ times.

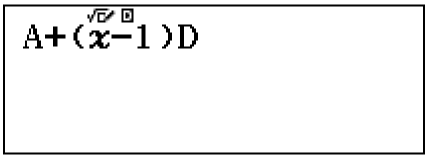
The first screen below shows how to set up the function in Table mode to generate terms of this arithmetic sequence. The second screen shows that a trip of 44 km would cost $182.



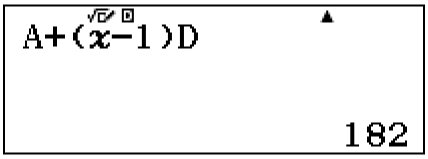


Notice in this case that the cost of a trip of 44 km is the 45th term of the sequence (as the first term is a trip of zero km).

Once again, individual terms of a sequence can be generated using CALC. In this case, three variables are needed for a, d and n. The calculator variables of A and D can be used, along with x (for n). Tap CALC and enter the values for the variables, starting with A = 6 and D = 4, followed by =.



Then use the ▼ key to highlight x and assign a suitable value, followed by =. The screens below show the earlier example of x = 45 to determine the fare for a 44 km trip. After assigning x = 45, tap = again to evaluate the 45th term:



The calculator retains previous values automatically, which makes this process fairly efficient. Tap = again to enter further values for x. Check for yourself that a 55 km trip (x = 56) will cost $226.

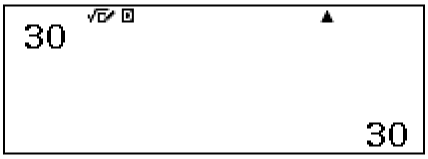
Geometric sequences

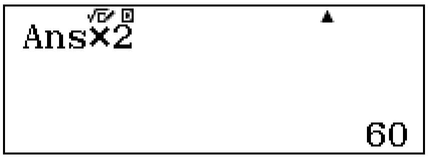
Geometric sequences are also important in mathematics. These differ from arithmetic sequences in the relationship between successive terms, which is multiplicative rather than additive. In general, the first term of a geometric sequence is represented by a and the *common ratio* between successive terms is represented by r. That is, each term can be found by multiplying the previous term by r.

An example in microbiology is the growth of the number of cells in a specimen. If there are originally 30 cells and the number doubles every hour, then a = 30 and r = 2. The sequence of the number of cells at the beginning of each hour is:

$$30, 60, 120, 240, 480, \ldots$$

This geometric sequence can be represented in the calculator either recursively or explicitly. The recursive version is summarised below.



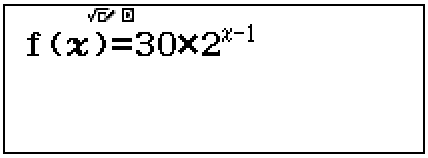


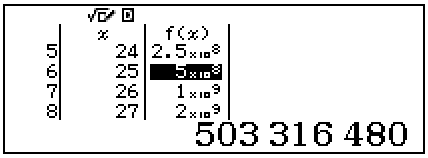
Tapping the = key will now generate further terms of the sequence. Although the numbers for the first few terms are easy to obtain in this way, it is quite inefficient to do this for later terms, such as finding out how many cells there will be after a full day's growth of 24 hours.

The explicit version uses the *n*th term of the geometric sequence, T_1, T_2, T_3, … with first term *a* and common ratio *r*, so the *n*th term is obtained by multiplying *a* by *r* a total of $n - 1$ times.

$$T_n = ar^{n-1}$$

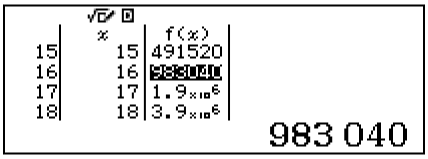
The first screen below shows how to set up the sequence in Table mode, while the second screen shows some terms of the sequence.

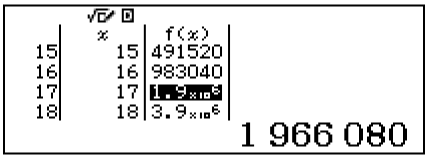




The second screen shows that after 24 hours have elapsed (that is, at the start of the 25th hour), there were more than 500 million cells in the specimen.

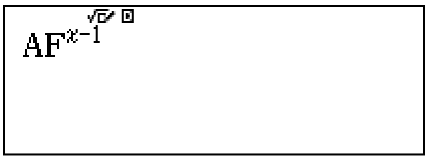
Having access to many terms of the sequence also allows other questions to be addressed easily. For example, suppose the scientist wanted to know when to expect there to be about one million cells.

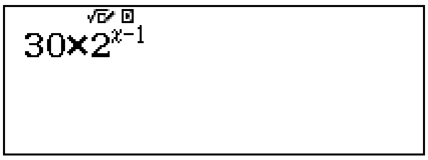




Remember that in general the *n*th term shows the population at the start of the $(n - 1)^{th}$ hour. These screens show that the 16th term of the sequence is almost one million, while the 17th term (an hour later) is almost two million. So it seems as if the cell population reaches one million shortly after 15 hours.

The terms of a geometric sequence can also be evaluated using CALC. In this case, a suitable generic expression is shown below, using *F* instead of *r* and *x* instead of *n*. An alternative is to use a particular version, such as that on the right below, which avoids a need to enter values for *a* and *r* repeatedly.





As previously, tap CALC to start the process of evaluating the expression.

Series

Some mathematical problems involve adding the successive terms of a sequence. The resulting sums are called a *series*. A famous example arises from the sum of the first n counting numbers, which gives the triangular numbers:

$$1 + 2 = 3$$

$$1 + 2 + 3 = 6$$

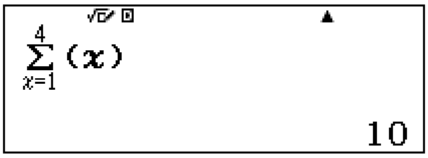
$$1 + 2 + 3 + 4 = 10, \text{ etc}$$

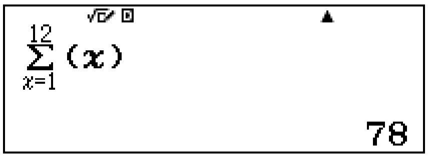
Each of these numbers is the sum of the terms of the sequence for which $T_x = x$. In mathematics, sums like these can be represented as follows:

$$\sum_{x=1}^{n} x$$

The Greek letter $\sum$ is the equivalent of a capital S and stands for 'Sum'. The limits below and above the $\sum$ show that the terms to be added start with the first term and finish with the nth term. In this case, the general term is simply x, for this sequence of counting numbers.

A series can be evaluated on the calculator, using the Σ key (via SHIFT x), as shown below, when an explicit definition for the general term is known. The general term of the sequence concerned must be described using x (i.e., with x) and the limits for the series entered as numbers as shown. Use ▶ to move from space to space.





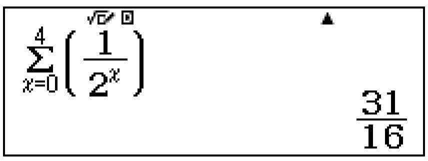
The screens show the 4th triangular number and the 12th triangular number. To get the second expression, it is a good idea to use the ◀ key to edit the first expression rather than starting again.

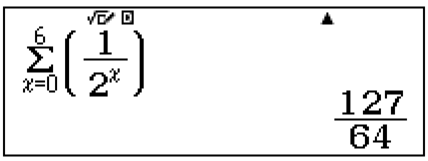
Many important series have an infinite number of terms, and so are not able to be represented entirely on the calculator. However, good approximations are available by summing a large number of terms.

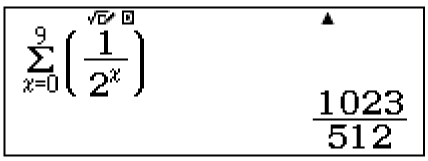
For example, consider the infinite geometric series:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ \ldots\ +\frac{1}{2^n}+\ \ldots\ = \sum_{0}^{n}\frac{1}{2^n}$$

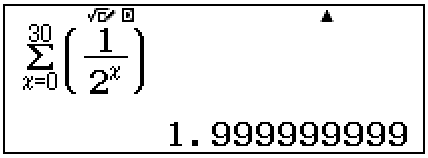
If successive terms of the series are obtained, the result gets closer and closer to 2, as the screens below suggest.

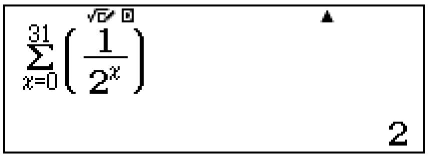




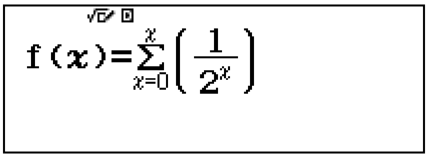
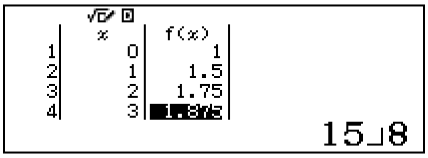


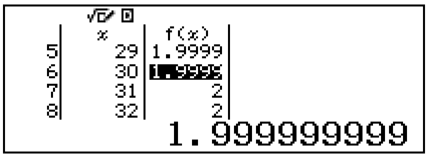
Although the total never quite reaches 2 (as each term covers half the remaining distance), the calculator regards it as close enough to 2 to round the result to 2 after 32 terms, as shown below.





Because an explicit term is known for the series, it can also be examined in a table, as shown below.



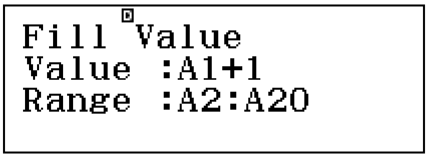
This sort of series is called *convergent* and is of considerable interest in higher mathematics.

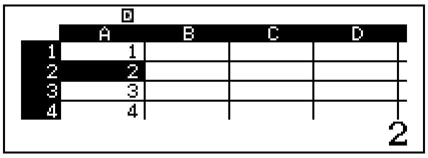
Sequences and series on spreadsheets

Both sequences and series can be defined and evaluated with a spreadsheet, through the use of suitable fill formulas. (See Module 13 for details of how to use spreadsheet fill formulas.) Whether an explicit or recursive definition for the *n*th term of a sequence is available, the terms of the sequence can be generated in a spreadsheet and the associated series also studied.

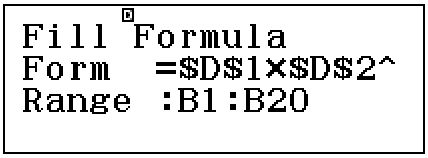
For example, a geometric sequence with general term $T_n = ar^{n-1}$ is represented in column B of the spreadsheet below, in which the first term *a* is in D1 and the common ratio *r* is in D2. Column A contains the values of the term number, *n*.

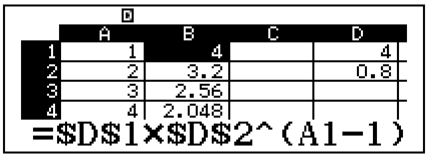
Start by entering 1 into A1 and then generating the first twenty values of *n*:



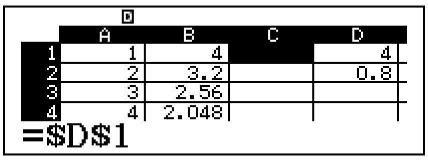


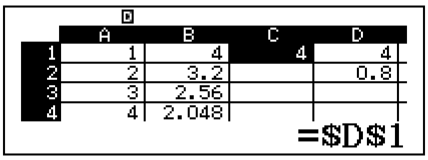
Then enter *a* into D1 and *r* into D2 and use a fill formula to generate the first twenty terms of the sequence. (The fill formula in B1 is shown in full in the second screen.) The particular case of the first twenty terms of the geometric sequence with *a* = 4 and *r* = 0.8 is shown here as an example:



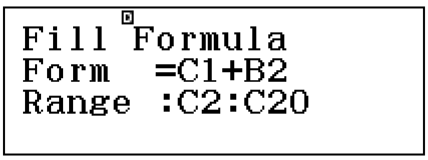


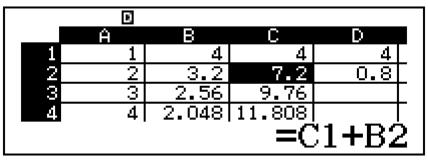
Finally, the associated series can be generated in column C. Start by storing the first term (of both the sequence and the series) into C1:



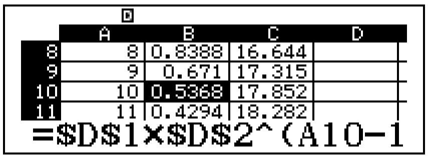


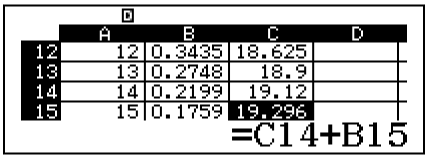
Then use a fill formula in C2 to add the new term of the sequence to the previous sum:



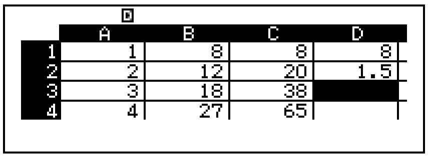


Once the sequence and the series are represented correctly, questions of interest can be answered by scrolling the spreadsheet. For example, the tenth term of the sequence is seen in B10 to be 0.5368, while the sum of the first 15 terms is seen in C15 to be 19.296. The series seems to be converging to 20 (although further terms might be needed to see this with more confidence.)





To study a different geometric sequence, such as that with $a = 8$ and $r = 1.5$, it is necessary only to adjust D1 and D2, as shown below. The new sequence and series are generated automatically.

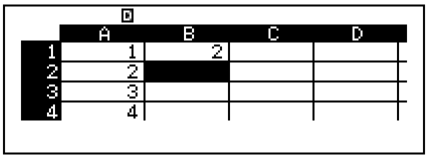


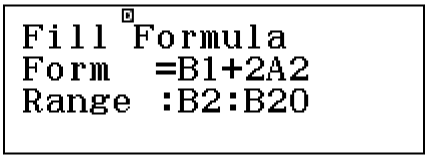
When an explicit definition is not known for a sequence, but a recursive definition is known, a spreadsheet is especially useful. For example, consider the sequence for which

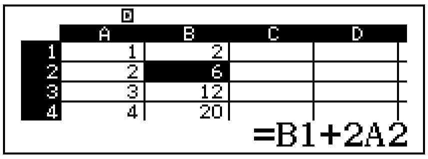
$$T_1 = 2$$
$$T_n = T_{n\text{-}1} + 2n, \text{ for } n > 1.$$

The second (recursive) step makes it clear how a term of this sequence can be generated from the previous term, but does not allow you to find a particular term of the sequence (such as the 15th term) or its associated series, so that tables, CALC and (Σ▬) commands cannot be used efficiently.

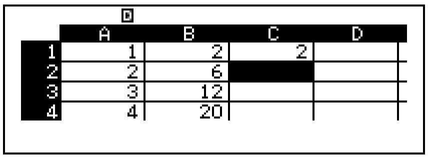
This sequence, and its associated series, can be represented in a spreadsheet by first entering entering 1 into A1 and then generating the first twenty values of n, exactly as was done above for the geometric sequence. Then enter the first term 2 into B1 and use the fill formula shown in B2 to generate the first 20 terms of the sequence in column B; the fill formula is derived directly from the recursive step in the definition of the sequence.

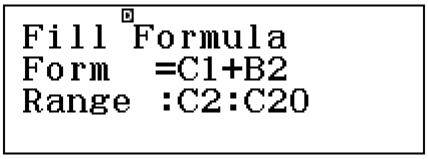


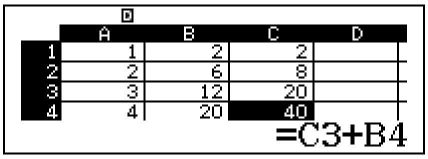




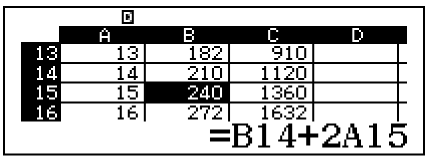
The associated series is generated in column C as before, also. Start by entering the first term in C1, and then each successive term is the sum of the new term and the previous sum:







Scroll to see the 15th term of the sequence is B15 = T_{15} = 240 and of the series is C15 = S_{14} = 1360.



After you see the patterns in the spreadsheet, you may see an *explicit* definition: $Tn = n(n + 1)$.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills

1. Use the calculator to count by threes, starting at 10.
2. Use the calculator to decrease by tens starting with 97.
3. A book costs \$32 today. If the price increases by 4% every year, set up a recursion to show the annual price. What will the price be in 12 years from now?
4. Generate the terms of this sequence recursively on the calculator:

   4, 12, 36, 108, 324, …
5. Use a table on your calculator to list the first five terms of the following sequences of numbers:

   (a) $T_n = 2n + 1$

   (b) $g_n = 3 \times 2^n$

   (c) $t_n = 80\left(\frac{1}{2}\right)^n$

   (d) $A_n = \dfrac{2n+1}{n+1}$
6. An arithmetic sequence has first term 12 and common difference 8. Use the calculator to make a table with the first thirty terms of this sequence. What is the twentieth term?
7. A geometric sequence has first term 20 and common ratio 0.95. Use the calculator to make a table with the first twenty terms of this sequence. What is the tenth term?
8. Edit the parameters for the sequence described in Exercise 7 to change the common ratio to 1.05 and to find the first 25 terms of the sequence. What is the tenth term?
9. Use the $(\Sigma\blacksquare)$ key to find the sum of the first ten squares: $1^2 + 2^2 + 3^2 + \ ... \ + 10^2$.
10. Edit the command you used for Exercise 9 to find the sum of the first 20 squares.
11. Use your calculator to find the sum of the first ten terms of:

    (a) 2 + 4 + 8 + …   (b) 0.03 + 0.06 + 0.12 + …
12. The second term of an arithmetic series is 15 and the fifth term is 21. Find the value of the common difference and the first term, then use your calculator to evaluate the sum of the first twelve terms.
13. Create a spreadsheet to find the first twenty terms of the arithmetic sequence 6, 11, 16, 21, … and of its associated sequence. Use the spreadsheet to find the twelfth term and the sum of the first 14 terms.
14. A sequence is stored in the first 30 cells of column A of a spreadsheet. Give the *Fill Formula* command that will store the associated series in column B.
15. Design a spreadsheet to represent the sequence: 1, 2, 3, 6, 11, 20, … in which each term after the third term is the sum of the preceding three terms. Find the 15^{th} term of the sequence.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. (a) Find the first term of the sequence 24, 8, $\frac{8}{3}$, $\frac{8}{9}$ that is less than 0.001. Which term is it?

   (b) Find the 20th term of the series 11 + 16 + 21 + 26 + … How many terms do you need to add in order to exceed a sum of 450?

   (c) A child has 100 building blocks and wants to build a triangular pile of blocks. She has one in the top row, two in the second row, three in the third row and so on. If she can use all of the blocks how many rows can she complete and how many will be left over?

2. Find the sum of the even numbers divisible by 3, lying between 400 and 500.

   (Use a table for $f(x) = 402 + (x - 1) \times 6$ to find the number of terms, then the (Σ▬) key to find the sum.)

3. (a) The expression $\sum_{1}^{n} x$ represents the sum of the first n counting numbers.

   Devise expressions to find the sum of the first n odd numbers and the first n even numbers.

   (b) Show that the sum of the odd numbers from 1 to 55 inclusive is equal to the sum of the odd numbers from 91 to 105 inclusive.

4. When a certain ball is dropped from a height of 3 m the first bounce takes 1 second (this is the interval between the instant that the ball hits the ground for the first and second times). Each subsequent bounce takes 2/3 of the time of the previous bounce.

   Find the total time taken for the first (i) 3 bounces (ii) 10 bounces (iii) 100 bounces.

   After how many seconds do you think the bouncing will have ceased?

5. Use the (Σ▬) key (via SHIFT x) or a spreadsheet to investigate the series associated with the powers of 2:

   $$1, 2, 4, 8, 16, 32, \ldots \text{ (i.e., } 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots)$$

   Can you find an easy way to determine the sum of the first k terms of this series?

6. As described in Module 6, the exponential function, e^x, can be defined by a remarkable infinite series:

   $$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

   (a) Use the (Σ▬) key or a spreadsheet to find the sum of the first eight terms for e^1. Then try some larger numbers of terms.

   (b) A calculator cannot give an infinite number of terms, so you should expect only an approximation to the result. How many terms are needed to get close to $e = 2.718281828\ldots$?

   (c) Use the series to investigate some other powers of e.

Notes for teachers

This module highlights the ways in which the *ClassWiz* can support students to think about sequences and series, and how these are related to the important idea of recursion. The module makes extensive use the inbuilt recursive capability of the calculator as well as Table mode and the special command for evaluating a series. Spreadsheets are helpful too. The text of the module is intended to be read by students and will help them to see how the calculator can be used to examine various aspects of these topics. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. 10, 13, 16, … [Use 1 0 = + 3 = =] 2. 97, 87, 77, …[Use 9 7 = − 1 0 = =] 3. $51.23 [Use 3 2 = × 1 . 0 4 = =] 4. Use 4 = × 3 = = 5. (a) 3, 5, 7, 9, 11 (b) 6, 12, 24, 48, 96 (c) 40, 20, 10, 5, 2.5 (d) 3/2, 5/3, 7/4, 9/5, 11/6 6. $T_{20} = 164$ 7. $T_{10} = 12.605$ 8. $T_{10} = 31.027$ 9. $\sum_{1}^{10} x^2 = 385$ 10. $\sum_{1}^{20} x^2 = 2870$ 11. (a) $\sum_{1}^{10} 2^x = 2046$ (b) $\sum_{1}^{10} 0.03 \times 2^{x-1} = 30.69$ 12. $d = 2$ and $a = 13$, $\sum_{1}^{12} (13 + 2(x-1)) = 288$ 13. $T_{12} = 61$, $S_{14} = 539$ 14. With T_1 in B1, in cell B2: Form =B1+A2, Range : B2:B30 15. 1, 2, 3 in A1, A2, A3. In A4, Form = A1+A2+A3, T_{15}=4841

Activities

1. The calculator allows students to address questions related to sequences and series directly rather than algebraically, but they may need some help to describe them using the calculator. [Answers: (a) Use a table with $f(x) = 24 \div 3x$ to find the 11th term (b) Use $\Sigma(6 + 5x)$ to see that 12 terms are needed (c) Use Σx to see that the sum of thirteen rows needs 91 blocks, so 9 are left over.]

2. This example is more sophisticated than those in Activity 1, as it requires both a table and the series command. You may choose to allow students to explore it for themselves at first without the substantial hint provided; in any event, a discussion is worthwhile about how the tabulated values represent the even numbers divisible by 3. [Answer 17 terms, adding to 7650.]

3. This activity requires students to think about how to represent a general term of a sequence. [Answers: (a) Assuming the series start with $x = 1$, suitable expressions are $\Sigma(2x - 1)$ and $\Sigma(2x)$ respectively, but different expressions are needed if the series starts with $x = 0$ (b) Use $\Sigma(2x - 1)$ with suitable limits to see that each sum is 784.]

4. Applications using infinite geometric sequences can be well approximated with a calculator, which provides a good sense of the rate of convergence by trying different limits in turn. In this case, the time for the x^{th} bounce can be represented as $(2/3)^{x-1}$ and the series obtained from $x = 1$. A good approximation to the sum to infinity is provided by taking a large number of terms. [Answers: (i) 19/9 seconds (ii) 2.948 secs (iii) 3 seconds (essentially the sum to infinity). A table or CALC shows 13th bounce takes less than 1/100th of a second, so the ball has essentially stopped.]

5. Encourage students to try some examples and look for a pattern, perhaps by listing results in a table on paper. The general term for the series is 2^x, but students will need to be careful to make sure that the limits start with zero. [Answer: the sum of k terms is 2^{k-1}, clear from a spreadsheet.]

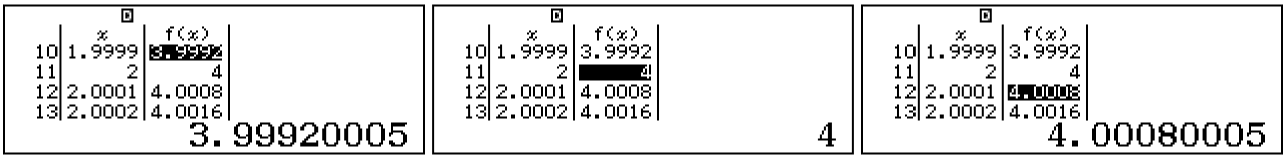
6. The expression for the series for e^1 is $1/x!$ but students need to be careful to start with $x = 0$. For efficiency, make sure that they tap ◀ to edit previous commands to get several results in succession. [Answers: (a) 2.71827877 (b) 12 terms are enough to get 2.718281828.]

Module 15
Calculus

The *ClassWiz* is very useful to represent and explore several aspects of introductory calculus. Several capabilities of the calculator are involved, including Calculate, Table and Spreadsheet modes. Make sure that the calculator is set to Math mode for input and output.

Continuity and discontinuity

Introductory calculus is mostly concerned with *continuous* functions. One way of thinking about continuous functions is that small changes in the variable are associated with small changes in the function itself. You can study this on your calculator using a table in which the values change only a small amount. There is an example below for the continuous function $f(x) = x^3 - x^2$ near $x = 2$.

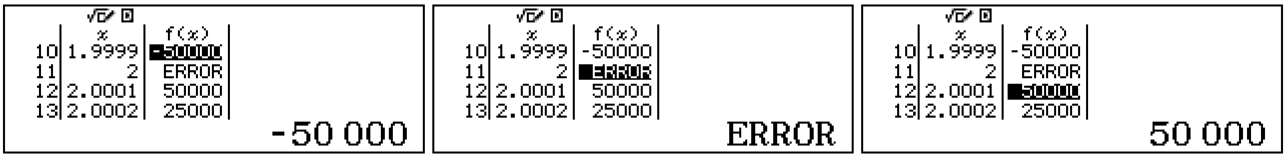


The screens suggest that the function is continuous at $x = 2$. Choosing smaller intervals for x will still result in small changes for $f(x)$ when the function is continuous.

When a function is not continuous, however, you will see that the values of the function can change dramatically with small changes in the variable. The example below shows the function

$$f(x) = \frac{5}{x-2}$$

near values for $x = 2$.

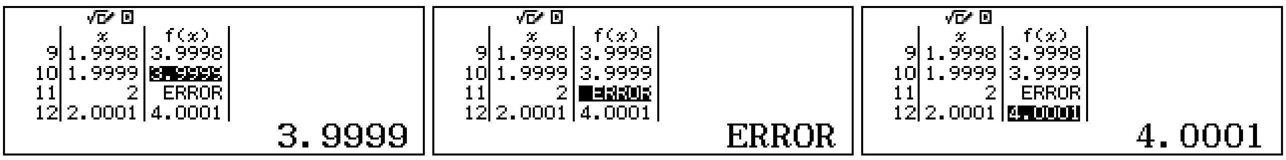


In this case, the values of the function jump from $f(1.999) = -50\ 000$ to $f(2.001) = 50\ 000$, a very large jump. In addition, the function is not defined for $x = 2$ (as shown by the error message). The table indicates that the function is discontinuous at $x = 2$. (This is sometimes called a *jump discontinuity*, as the values of the function 'jump' significantly for a small change in x.)

Some functions are discontinuous in other ways. For example, consider the function

$$f(x) = \frac{x^2 - 4}{x-2} \ .$$

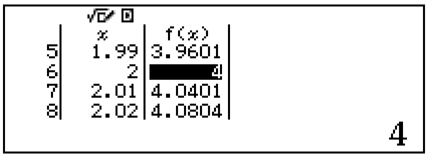
When x is close to 2, a table of values can allow you to explore values of the function:

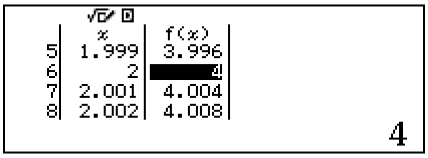


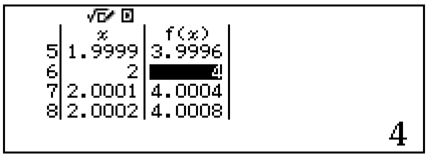
In this case, although the function is not defined for $x = 2$, the values of $f(x)$ do not jump dramatically on either side of $x = 2$. If you construct tables for the function on even smaller intervals, the same phenomenon will occur. In fact, for all values *except* $x = 2$, the function can be expressed as $f(x) = x + 2$. This kind of discontinuity is sometimes called a *removable discontinuity*, as it could be removed by defining a suitable value of the function at a point. In this case, if $f(2) = 4$, the function would be continuous.

Exploring the gradient of a curve

A major idea in the calculus concerns change. To study how a function is changing, a table of values is a powerful tool. For example, consider the function $f(x) = x^2$, when x is close to 2. The three screens below show how tables can be used to study the change near $x = 2$, by taking an increasingly small step. In the first table, the step is 0.01, in the second it is 0.001 and in the third table it is 0.0001





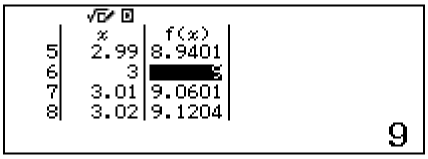


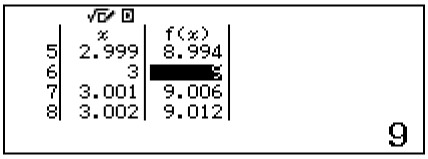
One way to approximate the rate of change of $f(x)$ at $x = 2$ is to consider the changes in the y-values divided by the changes in x-values on these small intervals. In turn, these three values are calculated below, using the values in the tables:

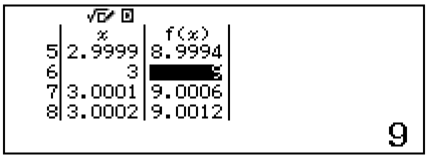
$$\frac{4.0401 - 3.9601}{0.02} \approx 4 \qquad \frac{4.004 - 3.996}{0.002} \approx 4 \qquad \frac{4.0004 - 3.9996}{0.0002} \approx 4$$

To the accuracy displayed in the tables, the rate of change is the same each time, with the values of $f(x)$ changing at about four times the rate of change of x. You could think of this as very close to a line with gradient of 4 near $x = 2$.

This function is changing differently in different places, as you can tell from imagining the graph of the function. To illustrate, consider the screens below which show the same function $f(x) = x^2$, when x is close to 3.







The associated gradients (the changes in y-values divided by the changes in x values) are:

$$\frac{9.0601 - 8.9401}{0.02} \approx 6 \qquad \frac{9.006 - 8.994}{0.002} \approx 6 \qquad \frac{9.0006 - 8.9994}{0.0002} \approx 6$$

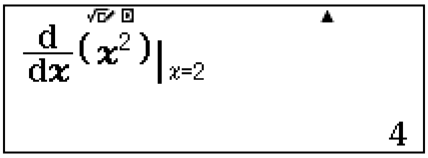
In this case, the gradient of the function at $x = 3$ seems close to 6. As expected, the gradient is larger when $x = 3$ than when $x = 2$, as the graph is steeper at that point.

You could continue to explore gradients for this function at various points using tables with decreasingly smaller intervals and decreasingly smaller step sizes, although it becomes a little tedious to do so. The smaller the interval, the more closely the gradient represents the rate of change of the function at a point, of course.

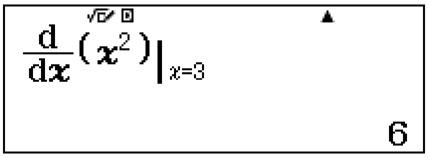
In the limit, as the size of the interval continues to decrease, the gradient of the curve at a point is defined as the *derivative* of the function at that point, which we will explore in the next section. In the case of the function $f(x) = x^2$, the gradient will always be given by the derivative function $f'(x) = 2x$. Notice that when $x = 2$, the gradient was $2x = 4$ and when $x = 3$, the gradient was $2x = 6$.

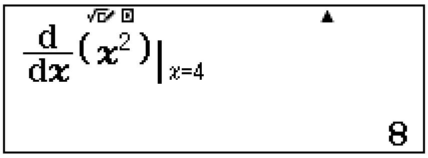
The derivative of a function

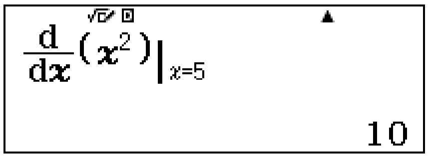
The calculator provides a special derivative key to obtain the numerical derivative of a function at a point, so that it is not necessary to analyse functions in detail using tabulated values as we did in the previous section. In Calculation mode (MENU 1), the derivative key ($\frac{d}{dx}$▮) (obtained with SHIFT ∫▭) is used for this purpose. To obtain the derivative of $f(x) = x^2$ at $x = 2$, start with ($\frac{d}{dx}$▮) and then enter the function rule and tap ▶ to enter a particular value of x (in this case 2), followed by =.



The result is the same as we obtained in the previous section. It is a *numerical* derivative, providing a numerical value at a single point. The general result, in this case that $f'(x) = 2x$, is not available on this calculator. To obtain the derivative of this function at other points, an efficient way is to tap the ◀ key and to then edit the command appropriately. In this way, you can quickly see several values such as those shown below:

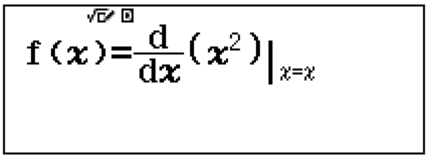


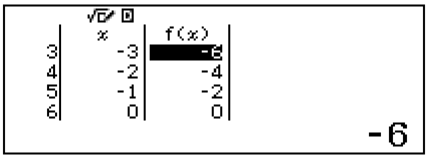


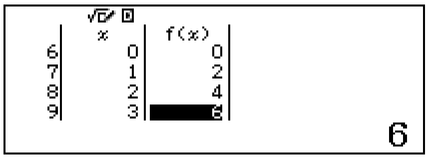


The derivative allows you to understand and describe the way in which the function is changing. In this case, as x increases, the derivative increases, so the function is increasing more rapidly and its graph becomes increasingly sloped upwards.

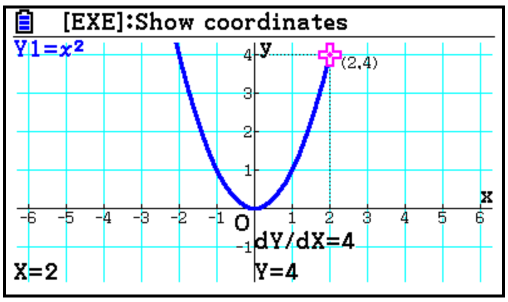
A table is a convenient way to examine several values, as shown below. If you explore the derivatives for negative values of x, you can see that the function is decreasing (i.e., has a negative slope) in a similar way (so that it is symmetric about $x = 0$). The derivative when $x = 0$ is zero, suggesting that the function is neither increasing nor decreasing at that point.







Obtaining values like these at various points helps to visualise a graph of the function, and how it is changing, like the one shown here, produced on a CASIO fx-CG20 graphics calculator.

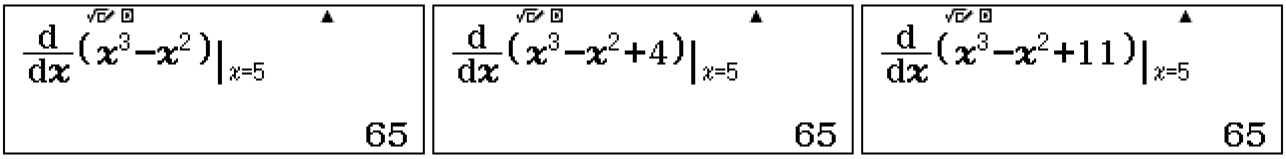


In this case, the graph can be seen to be parabolic in shape, symmetric about the y-axis and with a minimum point at the origin, all consistent with the observations made above. Notice that the graph is steeper at points further from the origin, as it is changing more quickly (positively and negatively on the two sides). As the calculator screens show, a common symbol for the derivative is dy/dx, and the graph screen shows that $dy/dx = 4$ when $x = 2$, as did the calculator screen earlier.

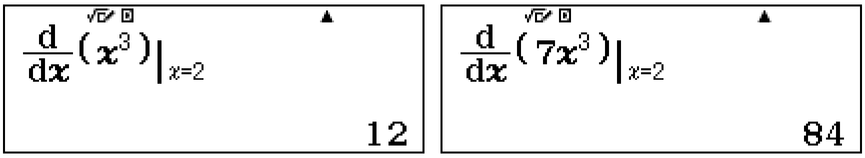
Properties of derivatives

You can use the derivative command to explore some properties of derivatives by examining their numerical values.

For example, the screens below suggest that adding a constant to a function does not change the value of the derivative. The derivative of $f(x) = x^3 - x^2$ at a particular point (e.g., $x = 5$) is not changed by the addition of constants to the function:

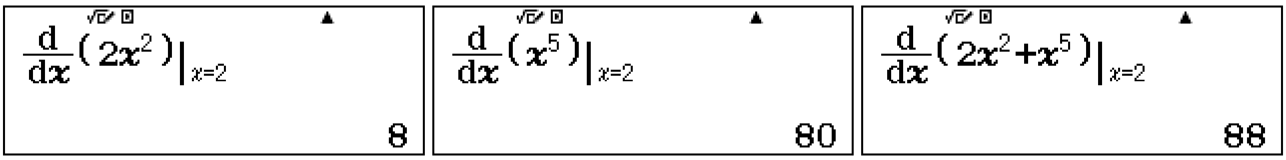


Similarly, multiplying a function by a constant has the same effect on the derivative. For example, the screens below show that multiplying the function $f(x) = x^3$ by 7 to get the new function $f(x) = 7x^3$ has the effect of multiplying the derivative at a particular point by 7 also:



In this case 7 x 12 = 84 shows the property. You can try this out with other constants (instead of 7) and at other points (than $x = 2$) to see that it is always true.

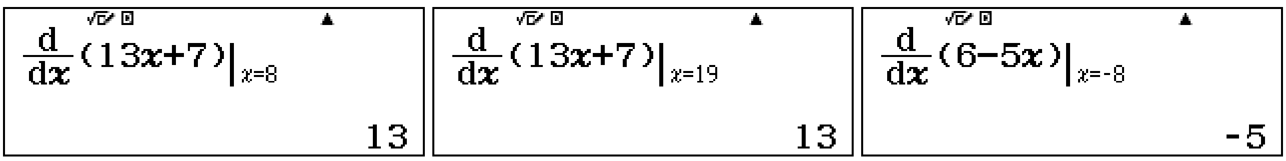
In a similar way, the derivative of a sum of two functions is the sum of their derivatives. The screens below show this for $f(x) = 2x^2$ and $g(x) = x^5$ at the point $x = 2$.



The derivative of the sum of these two functions is the sum of their two derivatives at the chosen point: 8 + 80 = 88. Try for yourself at some other points to see that this property holds in general.

You may have noticed that the derivative of a linear function is easily predicted from its slope, regardless of the point concerned: linear functions have the same derivative at *all* points, which is what gives them their characteristic shape of a line. The derivative is the slope of the line.

This relationship can be verified by using the derivative command, as shown below.

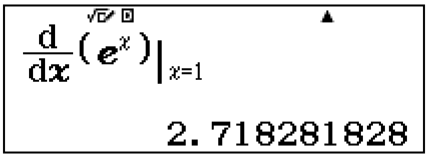


Although it is restricted to finding numerical values your *ClassWiz* will allow you to check for yourself many properties of functions and their derivatives. So, you will not be able to solve problems in general, or prove theorems about derivatives, but will be able to see for yourself many practical implications and solve practical problems using derivatives.

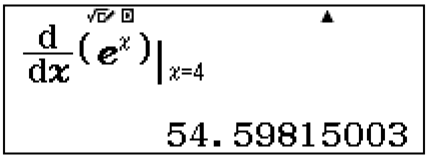
Two special derivatives

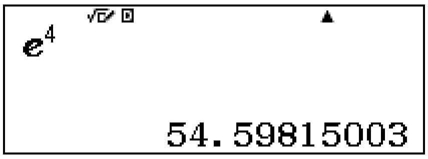
Derivatives allow us to imagine the shapes of graphs, as they describe how a function changes. The derivatives of some functions are especially surprising.

For example, consider the exponential function that was introduced in Module 6: $f(x) = e^x$. The derivative of this function at $x = 1$ is shown below:



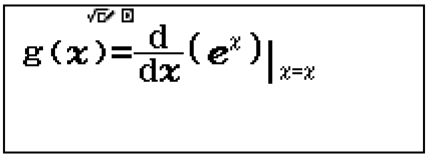
You probably recognise the value 2.718281828…, which is e itself. So, when $x = 1$, the derivative of $f(x) = e^x$ is the same as the value of the function. But the surprise continues with the following screen, for a different value of x ($x = 4$):



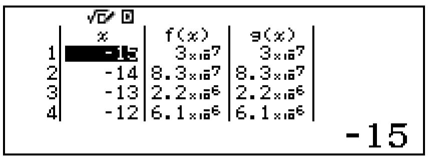


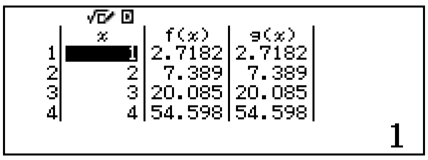
Once again, the derivative of the function is equal to the value of the function. Try this yourself for some other values of x to see that the exponential function has the extraordinary relationship that the function is its own derivative.

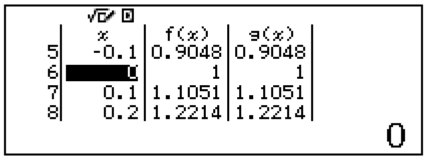
You can check this property with other values efficiently with a table or a spreadsheet. In the table below, for example, $f(x) = e^x$ and $g(x) = $ d/dx ($f(x)$):



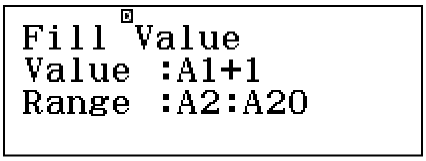
Regardless of the values for x, notice that $f(x) = e^x$ and its derivative $g(x)$ have the same value:

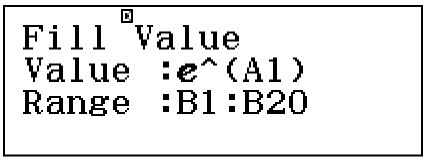


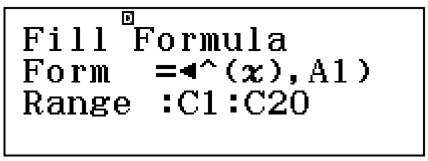




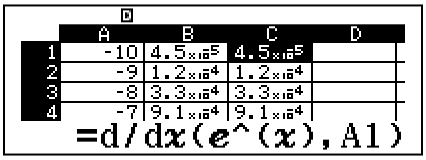
To study the derivatives with a spreadsheet, use column A for x, column B for $f(x) = e^x$ and column C for the derivative of $f(x)$, obtained as shown here:

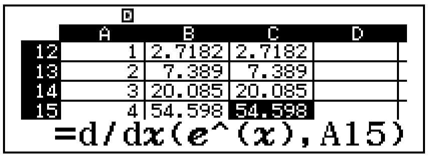


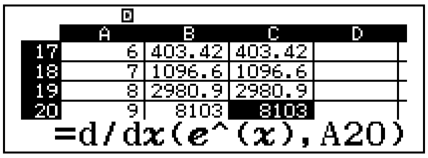




(It is unnecessary here to use fill formulas, and less memory is needed for fill values. This spreadsheet consumes about half the available space.) Notice especially that the fill formula in Column C (which can be seen below) has a different form (using a comma) from the natural display version of the derivative. The following screens show some values from this spreadsheet.



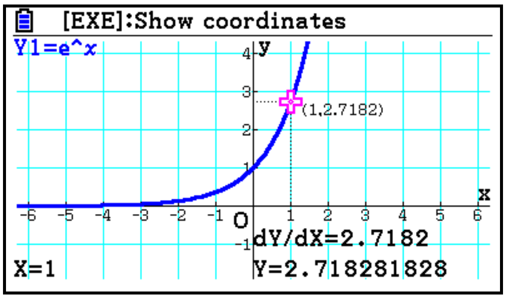




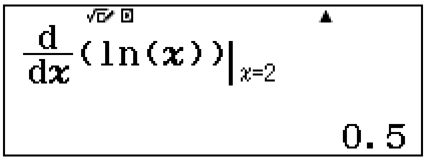
In fact, both the table and the spreadsheet suggest that this relationship holds for *all* values of the function: $f'(x) = e^x$. We can use this observation to imagine what the graph of the function will look like.

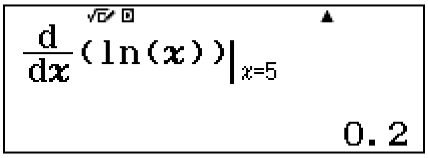
As the value of e^x is always positive, this observation about the derivative allows us to conclude that the function is always increasing. Since the value of e^x gets larger as x gets larger, then the exponential function $f(x) = e^x$ continues to increase at an ever-increasing rate as x increases, which is the essential idea of exponential growth, encountered in Module 6.

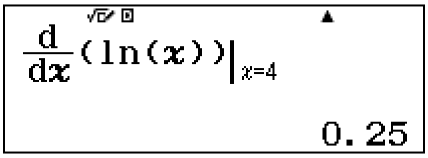
The graph of the exponential function here, generated by a CASIO fx-CG20 calculator, is consistent with these observations.



Related to the exponential function, the derivative of the natural logarithm function also produces an interesting pattern. Look carefully at the three screens below to see part of this pattern.

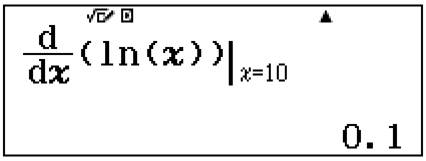


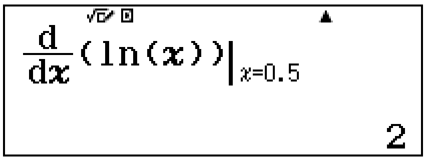


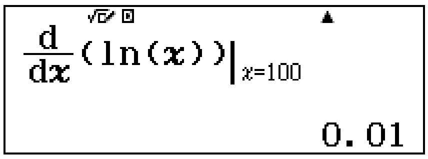


The derivative of the function $f(x) = \ln x$ seems to be the reciprocal of the value of x.

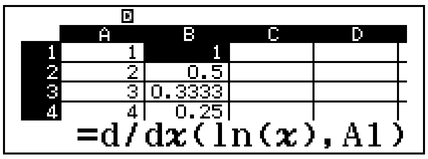
To check this property further, here are three further examples:

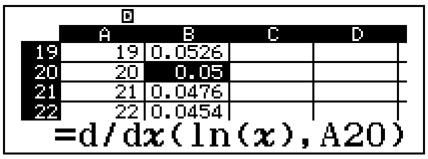


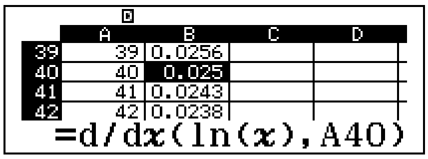




These examples follow the same pattern that the derivative of $\ln x$ is $1/x$. To see more values efficiently, a spreadsheet or a table is a useful tool.

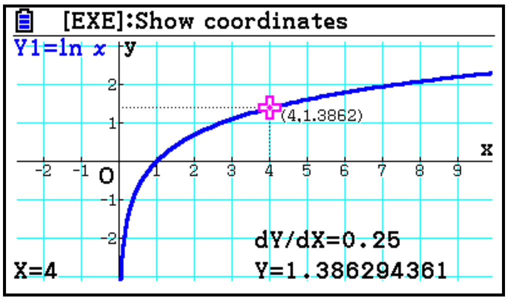






Observations like these about the values of the derivative suggest that the derivative of the natural logarithm function is always positive, and will help us to visualise what the graph will look like.

When x is a small value (such as $x = 0.01$), the derivative will be a large positive value and thus the graph will be very steep. As x increases, the derivative is still positive but decreases and quickly becomes almost (but not quite) zero, so that the graph of $f(x) = \ln x$ might be expected to 'flatten out' quickly after a steep beginning.



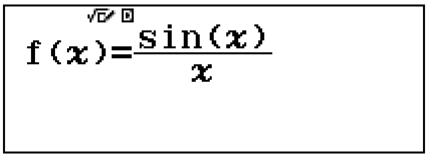
The graph shown here, generated by a CASIO fx-CG20 calculator, is consistent with these expectations.

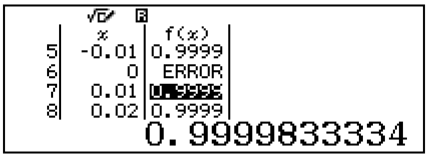
Exploring limits

An important idea in calculus concerns the *limit* of a function. Limiting values of some functions can be explored on your calculator using a table of values. A good example of this concerns the function

$$f(x) = \frac{\sin x}{x}$$

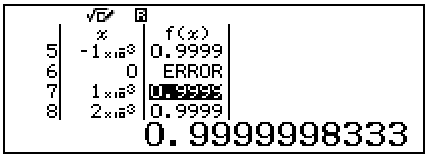
as x tends to 0. Notice that the function cannot be determined when $x = 0$ as both denominator and numerator are zero and division by zero is not defined. To explore this limit, we can construct a table of values as x gets close to zero. Make sure that the calculator is SET UP to radian mode. Define the function in Table mode and then evaluate it near zero:

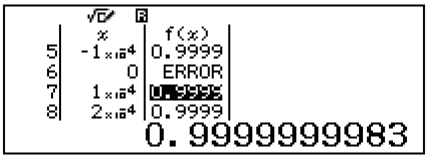


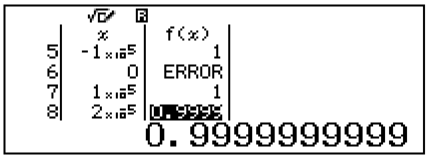


The screen above shows that the function is not defined for $x = 0$, but seems to have a value close to 1 for values of x near zero.

To consider the limiting situation further, choose increasingly smaller intervals by adjusting the table settings and scroll the values to observe how they change. The screens below show that, as x becomes extremely close to zero (i.e., when the *Step* is very small), the value of the function becomes extremely close to 1. Notice that each table gives an error for $f(0)$, which is undefined.

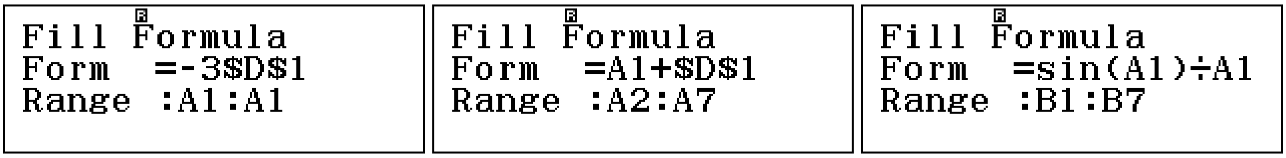




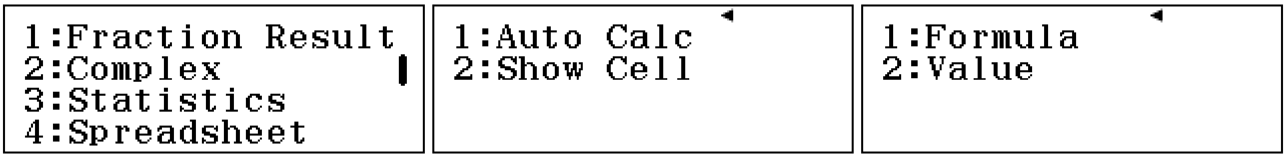


Indeed, in the final screen, the calculator displays a value in the table itself of $f(x) = 1$, as the best approximation to the actual value (which is not quite 1).

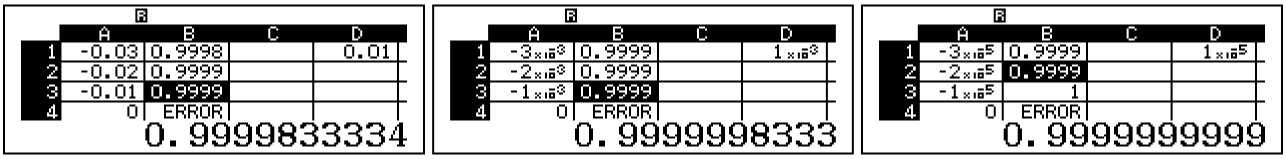
Another way to explore a limit of this kind is to use a spreadsheet. Rather than having to keep changing the *Step* of a table, you can design a spreadsheet to allow you to change the step more easily. Notice that the step is stored in cell D1 and the fill formulas in cell A1 and A2 use absolute addressing. (The $ signs are available with OPTN 1.) So, changing the value in D1 will allow you to get values for x in column A that are closer and closer to 0 and examine the value of the function in column B.



Finally, it's a good idea to now change the Show Cell setting in SET UP to show the Value instead of the Formula, as this will let you see more precise values than the spreadsheet column allows:



Here are some results of using this spreadsheet, by adjusting the value in D1 to get closer and closer to zero. (You will notice that each time you do this the spreadsheet reports a Math error, because of the division by zero in cell B4. Tap AC to proceed to see the spreadsheet.)



Scroll the spreadsheet to see how the values of the function change either side of $x = 0$. Notice again that the calculator gives a value of 1 eventually, as the best approximation to the result, which is a little less than 1.

We suggest that you change the Show Cell setting back to Formula when you have finished.

In the formal language of the calculus, the table and the spreadsheet show the important result

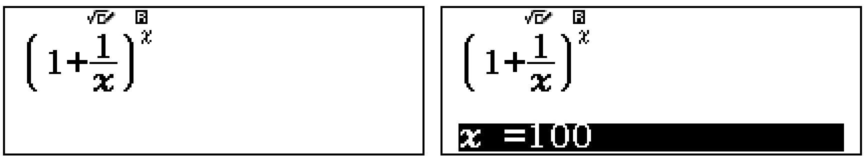
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Some other limits can be explored on the calculator in other ways. For example, the idea of a *limit at infinity* concerns the value approached by a function as the variable increases without bounds. As it is not possible to enter infinite values into functions, only good approximations can be obtained.

Consider a function used in Module 6 to describe aspects of exponential growth:

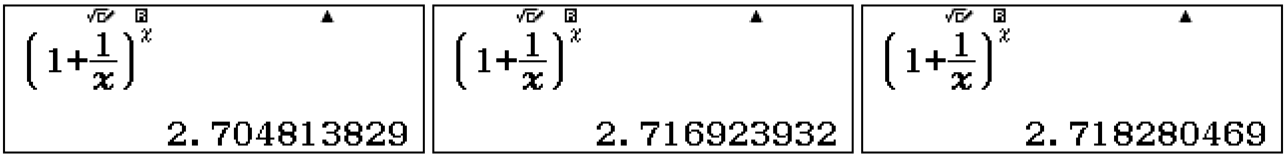
$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

To examine the limit approached by this function as x increases indefinitely, we can evaluate the function for increasingly large values of x. A good way to do this is to firstly define the function in Calculation mode (MENU 1) as shown on the left below (being careful to *not* tap =):

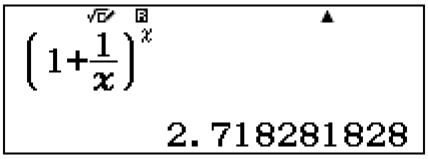


To evaluate the function as x increases, use the CALC command, choose a value for x and tap = to evaluate the function. Tap CALC again to continue.

Do this repeatedly, choosing increasingly larger values of x, to see the limiting process at work. The three values show $x = 100$, $x = 1000$ and $x = 1\ 000\ 000$ respectively:



The calculator has a practical limitation of screen size, but you can use processes like this to obtain a good approximation to a limit at infinity. In this case, the value of $x = 10\ 000\ 000\ 000$ gives a value as close to the limit as the screen will allow:



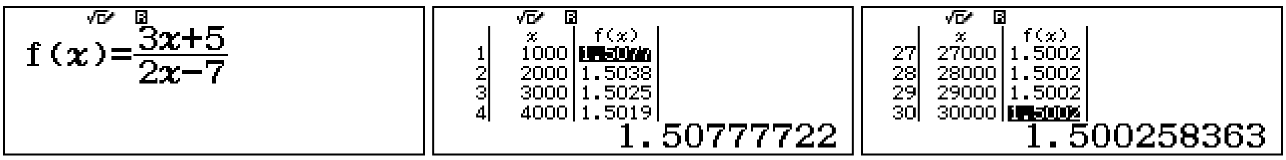
Of course, mathematical proofs are needed to establish limits, and 10 billion is a lot smaller than ∞, but the calculator can display good numerical approximations in these sorts of ways. In this case, the screen above reflects the very important mathematical result:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

As well as using the CALC facility, limits at infinity can also be explored using spreadsheets or tables of values. Choose very large values for the *End* values of a table (and a large *Step* as well to ensure that the table does not have too many entries). To illustrate this process, consider finding the limit as x tends to infinity of the rational expression:

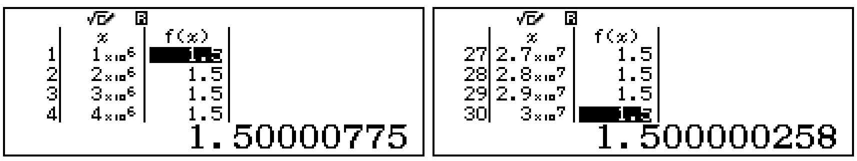
$$\frac{3x+5}{2x-7}$$

In Table mode, define a function that evaluates this expression and construct a table of values with very large values of x. For example, the screens below show a *Start* and *Step* of 1000 and an *End* value of 30 000:



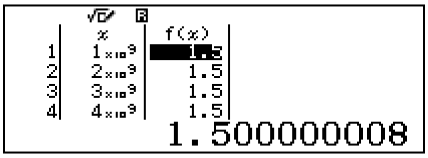
This table suggests that the limit is close to 1.5.

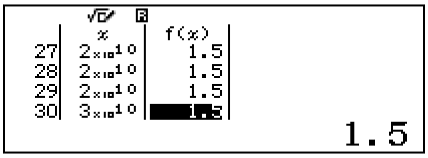
Choosing even larger values for the table parameters (one million for the *Step*) suggests the same result, even more clearly, as shown below:



Notice that the tabulated values are all shown on the screen as 1.5, because of the screen limitations, but the actual value is a little larger than this, as shown at the bottom of each screen.

Finally, the screen below shows the result of using one billion for the Step in the table, and suggests even more strongly that the limit value is 1.5. In this case, the value at the bottom of the screen is also showing as 1.5, because of the screen resolution, but in reality it is a little more than this.





A calculator result is not a proof, of course, but the result is consistent with the following statement that *can* be proved mathematically:

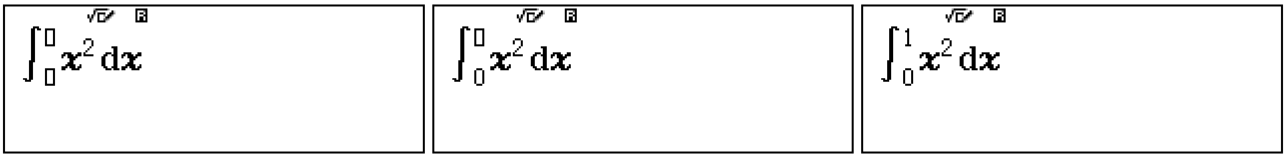
$$\lim_{x \to \infty} \frac{3x+5}{2x-7} = \frac{3}{2}.$$

Integration as area under a curve

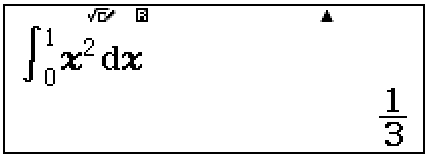
As well as differentiation, the calculator can be used to find the numerical value of definite integrals, using the $\int$ key. This is useful in many practical contexts, and can be regarded as finding the (signed) area under a graph of a function and above the x-axis.

To evaluate an integral numerically, start with the $\int$ key and then enter in turn the function (using x as the variable), the lower limit and the upper limit, tapping ▶ (or ◀) between values.

The following screens show the three successive steps to find the area under the graph of $f(x) = x^2$ and above the x-axis between $x = 0$ and $x = 1$.



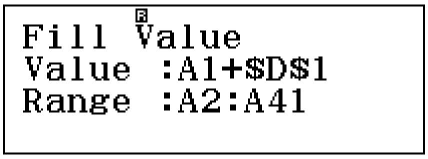
After the upper limit is entered, tap ≡ to evaluate the integral, in this case given as an exact value (unusually) below:

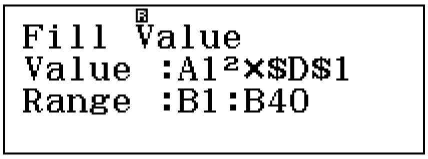


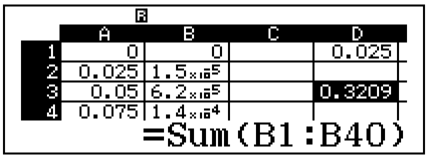
This is a numerical approximation to the area (which is why sometimes integrals take a few seconds to be evaluated on the calculator). To understand what is being calculated, consider the spreadsheet below to approximate a *Riemann sum* for this function. In column A, the interval from 0 to 1 has been divided into 40 equal intervals with A1 = 0 and A41 = 1. Each interval has the same width, which is shown in cell D1. Column B contains the areas of each of 40 thin rectangles under the curve; each has a width of D1. Adding the areas gives an approximation to the area under the curve.

There are several ways to find the areas of the rectangles, depending on how they are formed. One possibility is to use the *lower sum*, for which the height of each rectangle is the value of the function on the left end of an interval, which means that each rectangle is *below* the graph. The formula used is the formula for the area of a rectangle. Note that, for this function, A1^2 is the height of the first rectangle so the Fill Value formula uses A1^{2}x\$D\$1 for the area.

The total area is approximated (in cell D3) by adding all the small areas in column B. Study the screens below, which provide enough information for you to try this yourself.

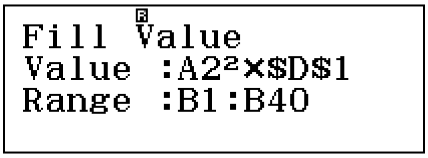


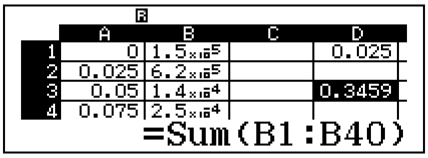




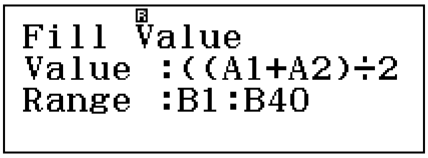
The resulting spreadsheet gives an approximation to the area of 0.3209 in cell D3. As expected, this value is a little too small.

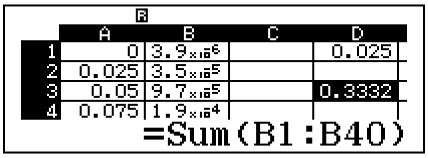
Another possibility is to use the *upper sum*, where each rectangle has a height equal to the value of the function at the right end of an interval, which means each rectangle is *above* the graph. Note that, for this function, $A2^2$ is the height of the first rectangle so the Fill Value formula uses $A2^2$x$D$1 for the area. You can edit the previous spreadsheet by changing the Fill Value formula in B1 to use this method. The resulting value of 0.3459 is a little too large.





A better possibility is to use a height equal to the height of the function $f(x) = x^2$ at the *mid-point* of the interval. The formula used is the formula for the area of a rectangle. (A1+A2)/2 is the midpoint of the interval: $((A1+A2)/2)^2$x$D$1. Again, you can edit previous spreadsheets to do this.

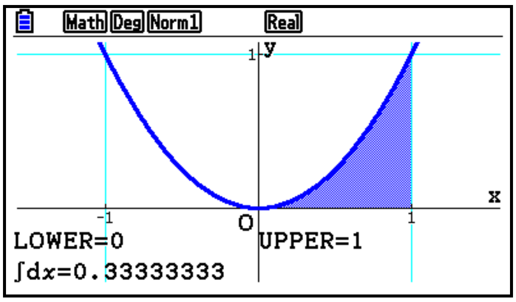




The result of 0.3332 obtained is between that for the lower sum and the upper sum, and closer to the value of 1/3 produced by the calculator.

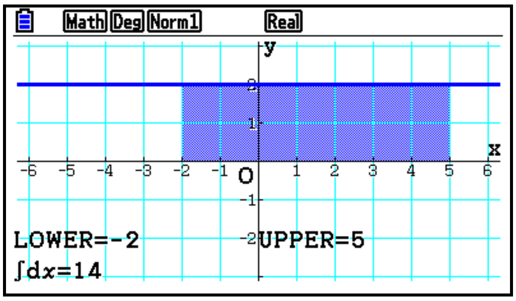
Notice that each of the areas in column B is very small for each method. A better approximation is obtained if even smaller intervals are used, but the spreadsheet is limited to 45 rows. The approximate area given by the spreadsheets is a little different from the figure of 1/3 given by the *ClassWiz*, but that is because the calculator uses many more intervals than 40 and continues refining the result until it is stable.

To understand what has been evaluated by this calculation, it is also helpful to see it graphically. The screen below from a CASIO fx-CG20 shows the shaded area under the graph of $f(x) = x^2$ that has been evaluated.

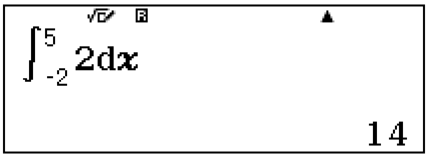


It is clear from the graph that this measurement of the area does indeed seem reasonable visually as it occupies less than half of the unit square in which it is located.

As another example, the screen below shows the integral of the constant function $f(x) = 2$ from $x = -2$ to $x = 5$. The area is that of a rectangle 7 units long and 2 units high, 14 square units.



On the calculator, the equivalent numerical result is readily obtained:



The rectangle example illustrates that the integral command allows you to explore areas of shapes defined by functions. More sophisticated examples than rectangles are possible.

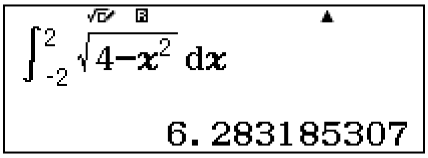
For example, draw a sketch for yourself to see that the area above the x-axis and under the line given by $f(x) = x + 1$ between $x = 1$ and $x = 4$ is a trapezium.

The area of the trapezium can be defined by an integral, as shown below:

A still more sophisticated example involves circular objects. Again, draw a sketch for yourself to see that the area under the graph of the following function between $x = -2$ and $x = 2$ defines a semicircle with centre at the origin and radius 2:

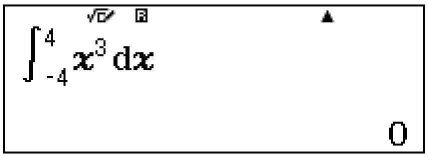
$$f(x) = \sqrt{4 - x^2}$$

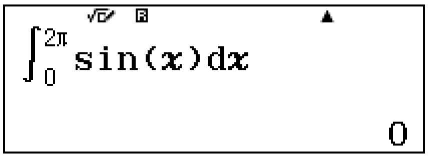
This is because the relation $x^2 + y^2 = 4$ defines a circle with radius 2 and centre at the origin. The area of this semicircle is given by the integral shown below. You will notice that this result takes several seconds to appear, as the calculator needs to make many calculations in order to obtain it to sufficient accuracy, using an iterative procedure:



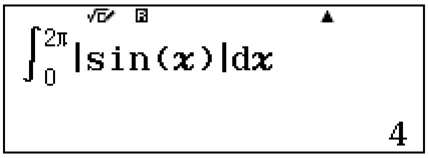
You will recognise this area as 2π, and so the area of the entire circle is 4π, thus verifying the relationship that the area of a circle is π times the square of its radius.

Care is needed in evaluating integrals numerically on the calculator, as the area calculated is a *signed* area. That is, an area below the horizontal x-axis is regarded as negative. For this reason, it is always wise to draw a sketch of a function before interpreting its integral. To understand this idea, check the following examples carefully by drawing the associated sketches for yourself:



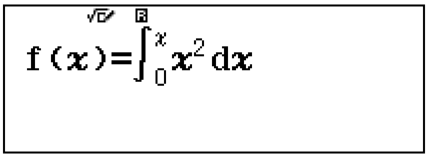


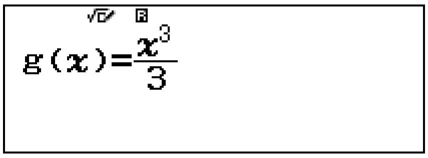
To find an area between the *x*-axis and a curve, use the absolute value function, which will have the effect of regarding areas below the axis as positive, as the example below shows:

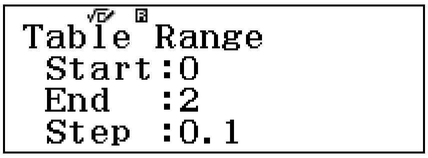


Another view of integration

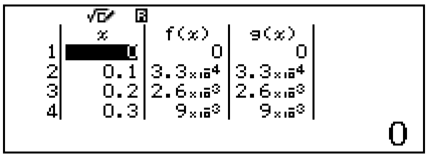
Another way of thinking about an integral is to consider the functions involved. To do this, we can study integrals from 0 to other values of *x*. Table mode is helpful here:

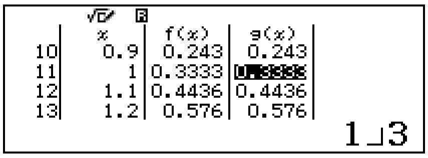


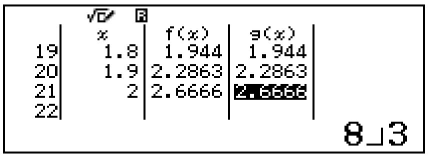




The extracts from the tables below show that the integral (in the $f(x)$ column) is the same as the function (in the $g(x)$ column), which helps to make clear how the integral and the result are related.

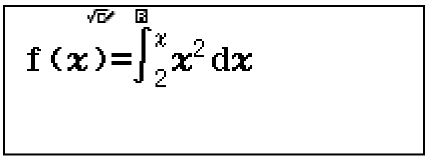


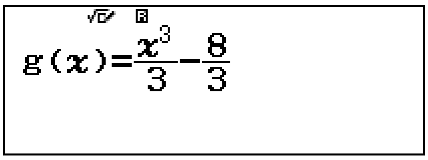


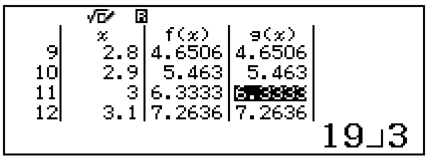


It seems in this case that $\int_0^x f(x)dx = g(x)$. Also, f(x) is the derivative of $g(x)$. That is $g'(x) = f(x)$. This illustrates that evaluating the integral of $f(x)$ can be handled by finding a function whose derivative is $f(x)$. This is usually called an *antiderivative* of f(x).

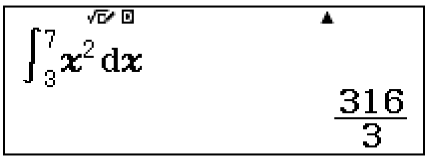
If the definite integral does not start at 0, the situation is a little more complicated. Study the example below, for which the definite integral starts at 2, and discuss it with others. Notice again that the two columns in the table have identical values.







These tables suggest that $\int_2^x f(x)dx = F(x) - F(2)$, where $F(x) = \frac{x^3}{3}$, the antiderivative of $f(x)$. So we could predict, say, $\int_3^7 f(x)\,dx = F(7) - F(3) = \frac{7^3}{3} - \frac{3^3}{3} = \frac{316}{3}$. This prediction is correct:



Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. (a) The function $f(x) = \dfrac{x-3}{5x-4}$ has a discontinuity on the interval $0 \le x \le 1$. Use a table of values to locate the discontinuity.
   (b) Is this a jump discontinuity or a removable discontinuity?

2. (a) Make a table of values of the function $f(x) = 2x + 5$ for $-10 \le x \le 10$ and a *step* of 1.
   (b) How can the table be used to describe the rate of change of the function?
   (c) Use the table to determine the derivative of the function at $x = 7$.

3. Use your calculator to obtain the numerical derivatives of the following functions for the given value of x:

   (a) $f(x) = x^2 + 2$ at $x = 3$
   (b) $f(x) = 4x^3 - 2x^2 + 1$ at $x = -1$
   (c) $f(x) = \dfrac{x+1}{2x-3}$ at $x = 5$
   (d) $f(x) = \dfrac{1}{x^2-4}$ at $x = 1$
   (e) $f(x) = \ln x$ at $x = 4$

4. (a) Use a table of values to evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

   (b) Use the CALC command to evaluate $\lim\limits_{x\to\infty} \dfrac{5-4x}{2x+3}$

5. Find the values of the numerical derivatives:
   (a) for $f(x) = x^2 + 3$ at $x = -0.5$, $x = 0$ and $x = 0.5$
   (b) for $f(x) = x^2 - 5$ at $x = -0.5$, $x = 0$ and $x = 0.5$
   Predict the values of the derivative of $f(x) = x^2 + 200$ at these same values and check your prediction with the calculator.

6. Use your calculator to evaluate the following numerical integrals:

   (a) $\int_0^1 x^2 dx$ (b) $\int_2^5 (3x-1)dx$ (c) $\int_1^5 (-x^2+4x+1)dx$

   (d) $\int_{-1}^4 (x^3-5x^2+2x+8)dx$ (e) $\int_{-1}^2 (x^3-5x^2+2x+8)dx$ and $\int_2^4 (x^3-5x^2+2x+8)dx$

7. (a) Find the area between the curve $f(x) = x^3 - 5x^2 + 2x + 8$ and the x-axis from $x = -1$ to $x = 4$. Compare your answer with Exercises 6(d) and 6(e).
   (b) Make a table of values of $f(x)$ from $x = -1$ to $x = 4$ in steps of 0.2.
   (c) Use your table to describe the changes in the function between -1 and 4.
   (d) Sketch the graph of $f(x)$ from $x = -1$ to $x = 4$.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some activities are too advanced for you. Ignore those you do not yet understand.

1. A table of values allows you to see how the values of a function $f(x)$ change as x changes. For a *linear function*, the change is the same for each interval.
(a) Tabulate $f(x) = x^2$, for $1 \le x \le 10$ with a *step* of 1 to confirm that f is *not* a linear function.
(b) Tabulate the function again on a small interval such as $1 \le x \le 1.001$ with a *step* of 0.0001; notice that the function is almost linear on this interval.
(c) Try some smaller intervals to see that the function is close to linear on very small intervals.
(d) Experiment with this idea with some other nonlinear functions.

2. A pharmaceutical company wishes to produce sealed cylindrical containers of thin metal with volume 25 cm^3 using the minimum amount of metal. If the inner radius of the container is r cm and the outside surface area is S cm^2, show that $S = 2\pi r^2 + \dfrac{50}{r}$.

   Use a table and numerical derivatives to find the value of r for minimum surface area.

3. A new housing complex is started with a population of 500 people.

   (a) It was planned that the population P would increase according to a model of $P = 500 + 100t$. What was the expected rate of change of the population each year?

   (b) The complex did not grow as planned and it was found that a better model was $P = 100(5 + t - 0.25t^2)$
   (i) What was the population after 1, 2 and 3 years?
   (ii) What was the rate of change of the population after 1, 2 and 3 years?
   (iii) What happened to the housing complex?

4. The derivative of a function can be used to explore the graph of a function near a turning point.
(a) For example, consider the parabola given by $f(x) = x^2 - 4x + 3$ near $x = 2$. Tabulate the derivatives using the derivative command on the calculator. How is the derivative changing around $x = 2$?
(b) Repeat for another quadratic function, such as $f(x) = 1 + 5x - x^2$ near $x = 2.5$.
(c) Examine derivatives of another quadratic function of your own choice near its turning point.
(d) Use your observations from (a) and (b) and a table of derivatives of $f(x) = \mathrm{x}^3 + 2x^2 - 3x + 1$ to find the (two) approximate turning points of the function. Check by drawing a graph of $f(x)$.

5. (a) Find the area between the x-axis and the curve given by $f(x) = x^3 - 3x^2 + 2x$ from $x = 0$ to $x = 1$, from $x = 1$ to $x = 2$ and from $x = 0$ to $x = 2$. How do these areas compare?
(b) Find the area between the x-axis and the curve given by $f(x) = x^3 - 6x^2 + 11x - 6$ from $x = 1$ to $x = 2$, from $x = 2$ to $x = 3$ and from $x = 1$ to $x = 3$. How do these areas compare?

6. Derivatives and integrals are related to each other, as you will study elsewhere in your course. The calculator allows you to explore some of these relationships. To do this, use a *ClassWiz* table of values to evaluate the following integral for several values of t, for $t > 1$:

$$\int_1^t \frac{1}{x}dx$$

   Notice that when $t = 2$, the integral is equal to ln 2. What pattern do you notice in your results?

   Use the second function in the table, that is $g(x)$, to check your predictions.

Notes for teachers

This module highlights the ways in which the *ClassWiz* can support students to learn about the calculus, particularly concepts of the derivative of a function, integrals, limits and continuity. The text of the module is intended to be read by students and will help them to see how the calculator can be used to deal with matrices in various ways. While the Exercises are best completed by students individually to develop calculator expertise, it is a good idea for students to do Activities with a partner, so that they can discuss their ideas together and learn from each other.

Answers to exercises

1. Jump discontinuity at $x = 0.8$ 2. Table changes by 2 as x increases by 1; derivative is 2 3. (a) 6 (b) 16 (c) -5/49 (d) -2/9 (e) ¼ 4. (a) 0 (b) 5/2 5. (a) -1, 0, 1 (b) -1, 0, 1 6. (a) 1/3 (b) 6 (c) 32/3 (d) 125/12 (e) 63/4 and -16/3 7. (a) 253/12 (63/4 + 16/3 or use absolute value function) (c) table shows roots at -1, 2 and 4, a maximum near $x = 0$ and a minimum near $x = 3$.

Activities

1. This activity addresses the key idea of *local linearity*: that most continuous functions are approximately linear on a small enough interval. The activity mimics zooming on a graph. The example suggested reveals the approximation involved, but smaller intervals will appear even more clearly to show the change in y-values to be twice the change in x-values. Encourage the students to try other functions as well to see that the result is not only relevant to quadratic functions.

2. Careful and repeated use of the calculator will allow students to explore tabled values of the function on various intervals to see approximately where the minimum surface area occurs. They can do this without knowing how to find the derivative symbolically. This will allow them to get close to $r = 1.5846$ cm to give $S \approx 47.33$ cm^2. If they find the derivative of S using the calculator, they will be restricted to three decimal places, but will be able to see that $S'(1.584)$ is negative and $S'(1.585)$ is positive, but each is close to zero. This should make for a good classroom discussion.

3. This activity is intended for students to explore a mathematical model using various calculator features to efficiently evaluate both functions and derivatives, interpreting results in context. [Answers: (a) 100 people per year (b) (i) 575, 600, 575 (ii) 50, 0, -50 (iii) the maximum population reached was 600, after which the population declined.]

4. By tabulating derivatives, students should find from the two parabolas that derivatives change in sign either side of a turning point and are zero at the turning point itself. This should suggest that they look for points where the derivative are zero as a key to finding turning points for a function such as that in part (c). [Answers: (a) $f'(x)$ changes from negative to positive at $x = 2$, and $f'(2) = 0$ (b) $f'(x)$ changes from positive to negative at $x = 2.5$, and $f'(2.5) = 0$ (d) turning points are at $x \approx -1.87$ and $x \approx 0.54$]

5. This activity is concerned with students realising that interpreting integrals as areas requires them to understand whether a graph crosses the x-axis. The graphs of the two functions cross the x-axis at the nominated points, so in each case two separate areas need to be found or an absolute value function used in the integral. Students might determine roots by factorising or through solving the associated equations (as described in Module 4). [Answers: (a) ¼, ¼, ½ (b) ¼, ¼, ½]

6. This activity is related to the development of the derivative of the natural logarithmic function in the module, and is concerned with the opposite property that the antiderivative of the reciprocal function is the natural logarithmic function. So the area under the curve is $\ln t - \ln 1 = \ln t$, which is a very surprising result. Some students might recognise the integral when $t = 2$ as $\ln 2 = 0.6931\ldots$ Encourage them to check a few values for t, including $t = e$. If necessary, suggest that they tabulate $g(x) = \ln x$ in the second column as well as the integral in the first column.

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